Alternative proofs of the asymmetric Lovász local lemma and Shearer's lemma

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Outline

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$E_i \leftrightarrow C_j$ is unsatisfied.

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$(0, 0, 1, 0, 1, 1)$: satisfying assignment.

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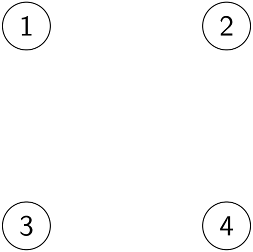

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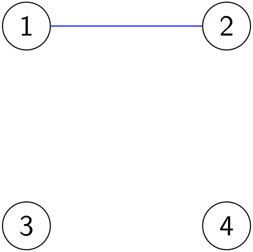

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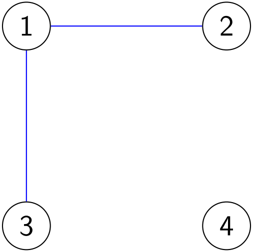

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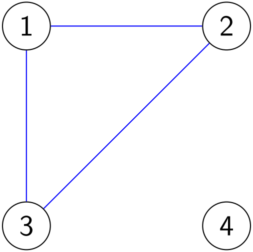

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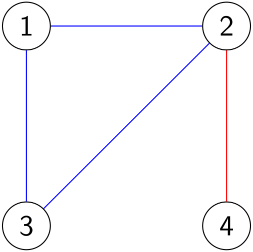

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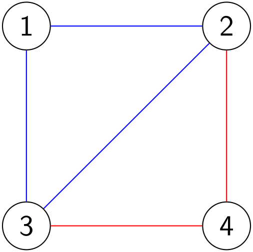

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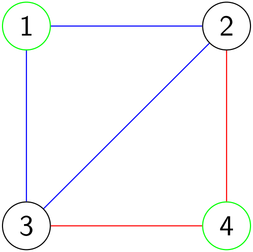

$\{1, 4\}$: independent set.

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▶ Moser and Tardos (2009-10): algorithmic proof of the symmetric (simple or lopsided) LLL.

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- Giotis, Kirousis, Psaromiligkos and Thilikos (2015): symmetric LLL.

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  - Use complementary "validation" algorithms to avoid dependencies introduced by the Moser-like algorithms.
- Using direct probabilistic arguments, express a bound to the probability that our algorithms last for at least n steps by a recurrence relation.
- Solve the recurrence by analytic means and show that this probability is inverse exponential in n.

- Giotis, Kirousis, Psaromiligkos and Thilikos (2015): symmetric LLL.
- Giotis, Kirousis, Livieratos, Psaromiligkos and Thilikos (2018): (variable-directed) lopsidependent LLL.

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Aim: Prove that P_n is inverse exponential in n.

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- $$P_n = \sum_{\mathcal{F}:|\mathcal{F}|=n} Pr[\text{M-ALGORITHM executes with witness forest } \mathcal{F}.]$$

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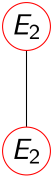
$$(0, 1, 0, 0, 1, 0)$$



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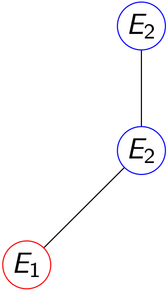

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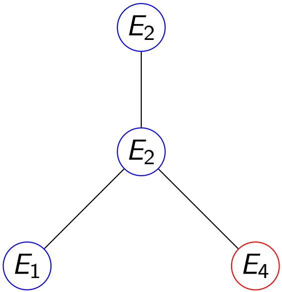

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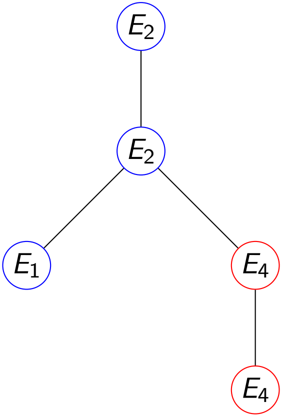

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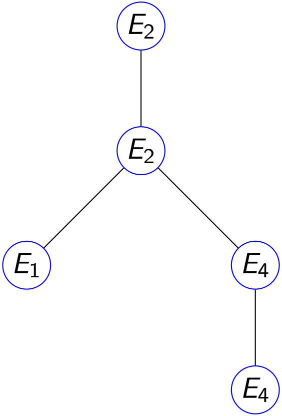

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$$P_n \leq \sum_{\mathbf{n}} \sum_{\mathbf{n}^1+\cdots+\mathbf{n}^m=\mathbf{n}} \left(Q_{\mathbf{n}^1,1} + R_{\mathbf{n}^1,1}\right) \cdots \left(Q_{\mathbf{n}^m,m} + R_{\mathbf{n}^m,m}\right).$$

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$$Q_{\mathbf{n},j} = Pr[E_j] \Big(Q_{\mathbf{n}-(1)_j,j} + R_{\mathbf{n}-(1)_j,j} \Big),$$

$$R_{\mathbf{n},j} = Pr[E_j] \sum_{\mathbf{n}^1 + \cdots + \mathbf{n}^{k_j} = \mathbf{n} - (1)_{m+j}} \Big(Q_{\mathbf{n}^1, j_1} + R_{\mathbf{n}^1, j_1} \Big) \cdots \Big(Q_{\mathbf{n}^{k_j}, j_{k_j}} + R_{\mathbf{n}^{k_j}, j_{k_j}} \Big).$$

- Solve $(Q_1(\mathbf{t}), \dots, Q_m(\mathbf{t}), R_1(\mathbf{t}), \dots, R_m(\mathbf{t}))$ by the multivariate Lagrange inversion formula [Bender and Richmond, 1998].

Outline

The Lemma

$G = (\{1, \dots, m\}, E)$ and $\bar{p} = (p_1, \dots, p_m)$.
$I(G) := \{\emptyset = I_0, I_1, \dots, I_s\}$ the set of independent sets of G.

$$q_I(G, \bar{p}) := \sum_{J \in I(G): I \subseteq J} (-1)^{|J \setminus I|} \prod_{j \in J} p_j.$$

For $E_1, \dots, E_m$ with dependency graph G and $Pr[E_j] = p_j$, $j = 1, \dots, m$:

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- I covers J if $J \subseteq N(I)$.

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- E_j does not depend on $E_t \in I_{i_t}$, thus it does not occur at the end of round t. **Contradiction**.

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2. Check each I_{i_t}. **If** there is a non-occurring E_j in I_{i_t}, `fail`. **Else**, resample vbl(I_{i_t}).

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2. Check each I_{i_t}. **If** there is a non-occurring E_j in I_{i_t}, `fail`. **Else**, resample vbl(I_{i_t}).
3. If P has no other nodes, GenVal `succeeds`.

Witness Paths

P: path with n nodes, labeled by $I_{i_1}, \ldots, I_{i_n}$.

$$\mathbb{P}_n := Pr[\textsc{GenResample} \text{ lasts for at least } n \text{ rounds}]$$
$$= \sum_{P:|P|=n} Pr[\textsc{GenResample} \text{ executes with path } P].$$

<u>GenVal</u>

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$$\mathbb{P}_n \leq \sum_{P:|P|=n} Pr[\textsc{GenVal} \text{ succeeds with path } P].$$

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$$Pr[V_P] := Pr[\textsc{GenVal} \text{ succeeds on input P}].$$

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- Show that Q_{n_i} are inverse exponential to n.

Outline

Stable set matrix

- M $s \times s$ matrix, $M(i,j) = \prod_{j \in I} p_j$ if I covers J. Else it is 0.

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$$\mathbb{P}_n \leq \sum_{i=1}^{s} Q_{n,i} = ||q_n||_1,$$

where $|| \cdot ||_1$ is the 1-norm.

Matrix norms

$$||M||_1 := sup_{x \neq 0} \frac{||Mx||_1}{||x||_1} \geq \frac{||Mq_1||_1}{||q1||_1},$$

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- $p(M)$ spectral radius of M.

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Thus:

$$||M^{n-1}||_1 < (p(M) + \epsilon)^{n-1}.$$

Thank you!