q-Random Walks on the Integers and on the Two-Dimensional Integer Lattice

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q-Stories: Selected References

- S. Janson, Y.C. Stamatiou and M. Vamvakari, Bounding the unsatisfiability threshold of random 3-SAT, *Random Structures and Algorithms*, 17 (2000), 103-116.
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q-Series Preliminaries, $0 < q < 1$

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- The q-binomial coefficient is defined by

$$\binom{n}{k}_q := \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} = \frac{[n]_q!}{[k]_q![n-k]_q},$$

where

$$[n]_q! = [1]_q[2]_q \dots [n]_q = \frac{(q;q)_n}{(1-q)^n} = \frac{\prod_{k=1}^{n}(1-q^k)}{(1-q)^n}$$

is the q-factorial number of order n with $[t]_q = \frac{1-q^t}{1-q}$.

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is the q-factorial number of order n with $[t]_q = \frac{1-q^t}{1-q}$.

- The q-binomial coefficient $\binom{n}{k}_q$, for n and k positive integers, equals the k-combinations $\{m_1, m_2, \dots, m_k\}$ of the set $\{1, 2, \dots, n\}$, weighted by $q^{m_1+m_2+\cdots+m_k-\binom{k+1}{2}}$,

$$\sum_{1 \le m_1 < m_2 < \cdots < m_k \le n} q^{m_1+m_2+\cdots+m_k-\binom{k+1}{2}} = \binom{n}{k}_q.$$

q-Series Preliminaries, $0 < q < 1$

- Let n be a positive integer and let x, y and q be real numbers, with $q \neq 1$. Then, a version of q-Cauchy formula is

$$\binom{x+y}{n}_q = \sum_{k=0}^{n} q^{k(y-n+k)} \binom{x}{k}_q \binom{y}{n-k}_q .$$

q-Series Preliminaries, $0 < q < 1$

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$$\binom{x+y}{n}_q = \sum_{k=0}^{n} q^{k(y-n+k)} \binom{x}{k}_q \binom{y}{n-k}_q .$$

- The q-multinomial coefficient is defined by

$$\binom{n}{k_1, k_2, \ldots, k_{r-1}}_q = \frac{[n]_q!}{[k_1]_q![k_2]_q! \cdots [k_{r-1}]_q![k_r]_q!},$$

where $k_r = n - k_1 - k_2 - \cdots - k_{r-1}$, for $k_i = 0, 1, 2, \ldots, n$, $i = 1, 2, \ldots, r, n = 0, 1, 2, \ldots$.

q-Binomial Distribution

q-Binomial Distribution

- Let a sequence of q-Bernoulli trials with varying probability of success at the ith trial,

$$p_i = \frac{\theta q^{i-1}}{1+\theta q^{i-1}}, i = 1, 2, \ldots,\ 0 < q < 1,\ 0 < \theta < \infty.$$

q-Binomial Distribution

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$$p_i = \frac{\theta q^{i-1}}{1+\theta q^{i-1}}, i = 1, 2, \ldots, \ 0 < q < 1, \ 0 < \theta < \infty.$$

- Then the probability function (p.f) of the number X of successes at n such trials is given by

$$f_X(x) = P(X = x) = \binom{n}{x}_q \frac{q^{\binom{x}{2}} \theta^x}{\prod_{j=1}^{n}(1+\theta q^{j-1})}, \ x = 0, 1, \ldots, n,$$

for $\theta > 0$, $0 < q < 1$. The distribution of the random variable (r.v.) X is called q-binomial distribution, with parameters n, θ and q (see Charalambides (2016)).

q-Binomial distribution: q-mean, q-variance

- The mean value, say μ_q, of the random variable $Y = [X_n]_{1/q}$ is

$$\mu_q = [n]_q \frac{\theta}{1 + \theta q^{n-1}}$$

q-Binomial distribution: q-mean, q-variance

- The mean value, say μ_q, of the random variable $Y = [X_n]_{1/q}$ is

$$\mu_q = [n]_q \frac{\theta}{1+\theta q^{n-1}}$$

- The variance, say σ_q^2, of the r.v. Y is

$$\begin{aligned}\sigma_q^2 &= \frac{\theta^2[n]_q[n-1]_q}{q(1+\theta q^{n-1})(1+\theta q^{n-2})} + \frac{\theta[n]_q}{(1+\theta q^{n-1})} \\ &- \frac{\theta^2[n]_q^2}{(1+\theta q^{n-1})^2}\end{aligned}$$

q-Asymptotics: q-Stirling Formula

q-Asymptotics: q-Stirling Formula

Theorem (Kyriakoussis and Vamvakari(2013))

The q-Stirling formula for $n \to \infty$, of the q-factorial number of order n, is given by

$$[n]_q! \quad \cong \quad \frac{(2\pi(1-q))^{1/2}}{(q \log q^{-1})^{1/2}} \frac{q^{\binom{n}{2}} q^{-n/2} [n]_{1/q}^{n+1/2}}{\prod_{j=1}^{\infty} (1 + (q^{-n} - 1) q^{j-1})}.$$

q-Asymptotics: q-Analogue of De-Moivre Laplace Theorem

Theorem (Kyriakoussis and Vamvakari(2013))

Consider the probability function (p.f.) of the q-binomial distribution, where $\theta = \theta_n,\ n = 0, 1, 2, \ldots$ such that $\theta_n = q^{-\alpha n}$ with $0 < a < 1$ constant. Then, for $n \to \infty$, the q-binomial distribution is approximated by a deformed standardized continuous Stieltjes-Wigert distribution as follows:

$$\begin{aligned} f_{X_n}(x) \quad &\cong \quad \frac{q^{-7/8}}{\sigma_q(2\pi)^{1/2}} \left(\frac{\log q^{-1}}{q^{-1}-1}\right)^{1/2} \left(q^{-3/2}(1-q)^{1/2}\frac{[x]_{1/q}-\mu_q}{\sigma_q} + q^{-1}\right)^{-1/2} q^{-x} \\ &\cdot \quad \exp\left(\frac{1}{2\log q}\log^2\left(q^{-3/2}(1-q)^{1/2}\frac{[x]_{1/q}-\mu_q}{\sigma_q} + q^{-1}\right)\right), \quad x \geq 0, \end{aligned}$$

where μ_q and σ_q are the q-mean and q-variance of the q-binomial distribution.

A q-Random walk on the integers

A q-Random walk on the integers

Definition

The Markov chain whose state space is the set of all integers, and whose transition probabilities are given by

$$P_{i,i+1} = \frac{\theta q^{i-1}}{1+\theta q^{i-1}} \quad = \quad 1 - P_{i,i-1}, i = 0, \pm 1, \pm 2, \ldots,$$
$$0 < q < 1, \ 0 < \theta < \infty$$

is called q-random walk on the integers.

A q-Random walk on the integers

A q-Random walk on the integers

The probability that the chain will be back in initial state after $2n$ transitions, $n = 0, 1, 2, \ldots$, if n of them were increases and n of them were decreases , is the q-binomial probability

$$P_{0,0}^{2n} = \begin{pmatrix} 2n \\ n \end{pmatrix}_q \frac{q^{\binom{n}{2}} \theta^n}{\prod_{j=1}^{2n}(1 + \theta q^{j-1})}, \ \theta > 0, \ 0 < q < 1.$$

A q-Random walk on the integers

The probability that the chain will be back in initial state after $2n$ transitions, $n = 0, 1, 2, \ldots,$ if n of them were increases and n of them were decreases , is the q-binomial probability

$$P_{0,0}^{2n} = \binom{2n}{n}_q \frac{q^{\binom{n}{2}} \theta^n}{\prod_{j=1}^{2n}(1+\theta q^{j-1})}, \ \theta > 0, \ 0 < q < 1.$$

Theorem

The q-random walk on the integers after $2n$ steps, starting from the origin 0, for θ constant or for $\theta = q^{-\alpha n}$, $0 < a < 1$, $n = 0, 1, 2, \ldots$ is recurrent, while for $\theta q^n \to \infty$, as $n \to \infty$, is transient.

A q-Random Walk Stochastic Process

A q-Random Walk Stochastic Process

- Consider the time interval $(0, t]$, $t > 0$, and partition it into parts which are geometrically decreasing with rate q, defined by

$$\delta_i(n; t) = (n)_q^{-1} q^{i-1} t, \;\; i = 1, 2, \ldots, n, \;\; n \geq 1.$$

A q-Random Walk Stochastic Process

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$$\delta_i(n; t) = (n)_q^{-1} q^{i-1} t, \;\; i = 1, 2, \ldots, n, \;\; n \geq 1.$$

- Consider the process generated by making a step of length δ to the right and a step of length δ to the left at every time period $(n)_q^{-1} q^{i-1} t$, $i = 1, 2, \ldots, n$, with probability of success (right step) and probability of failure (left step) given by

$$P\left(X_i = \delta\right) = \frac{\theta(n)_q^{-1} q^{i-1} t}{1 + \theta(n)_q^{-1} q^{i-1} t} \text{ and } P\left(X_i = -\delta\right) = \frac{1}{1 + \theta(n)_q^{-1} q^{i-1} t},$$

where $0 < q < 1, 0 < \theta < \infty$.

A q-Random Walk Stochastic Process

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$$\delta_i(n;t)=(n)_q^{-1}q^{i-1}t,\ \ i=1,2,\dots,n,\ \ n\geq 1.$$
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$$P\left(X_i=\delta\right)=\frac{\theta(n)_q^{-1}q^{i-1}t}{1+\theta(n)_q^{-1}q^{i-1}t}\text{ and }P\left(X_i=-\delta\right)=\frac{1}{1+\theta(n)_q^{-1}q^{i-1}t},$$
where $0<q<1, 0<\theta<\infty$.
- At time $\sum_{i=1}^{n}\delta_i(n;t)=t$, the position of the process is the r.v.
$$X_{n,q}(t)=\sum_{i=1}^{n}X_i.$$

Definition

The continuous time stochastic process $\{X_{n,q}(t), t \geq 0\}$, is called *q-random walk stochastic process* with parameters q, n and θ,$0 < \theta < \infty$ if the following properties hold

(a) In each of the consecutive mutually disjoint time intervals of length $\delta_i(n;t) = (n)_q^{-1} q^{i-1} t$, $i = 1, 2, \dots, n$, $n \geq 1$, at most one event (right or left step of length $\delta = 1$) occurs and

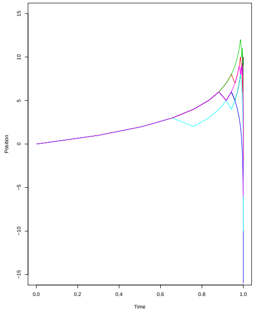
$$P\left(X_{n,q}\left(\frac{1-q^i}{1-q^n}t\right) - X_{n,q}\left(\frac{1-q^{i-1}}{1-q^n}t\right) = \delta\right) = \frac{\frac{\theta q^{i-1} t}{(n)_q}}{1 + \frac{\theta q^{i-1} t}{(n)_q}},$$

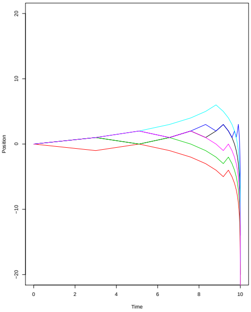
$$P\left(X_{n,q}\left(\frac{1-q^i}{1-q^n}t\right) - X_{n,q}\left(\frac{1-q^{i-1}}{1-q^n}t\right) = -\delta\right) = \frac{1}{1 + \frac{\theta q^{i-1} t}{(n)_q}}.$$

(b) The increments $X_{n,q}(t_i) - X_{n,q}(t_{i-1})$, where $t_i - t_{i-1} = (n)_q^{-1} q^{i-1} t$, $i = 1, 2, \dots, n$ are independent.

(c) The process starts at time $t = 0$ with $X_{n,q}(0) = 0$.

q-Random Walk Simulations in R





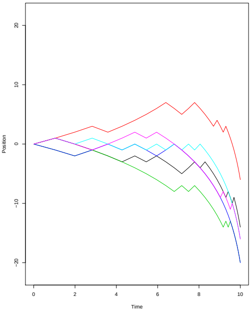


Figure 1: Generation of 5, q-Random Walk Processes in R with with $q = 0.7$, $n = 30$, $\theta = q^{-15}$, $t = 1$.
Figure 2: Generation of 5, q-Random Walks Processes with $q = 0.7$, $n = 30$, $\theta = 1$, $t = 10$.
Figure 3: Generation of 5, q-Random Walks Processes with $q = 0.9$, $n = 30$, $\theta = 1$, $t = 10$.

q-Random Walk Simulations in R

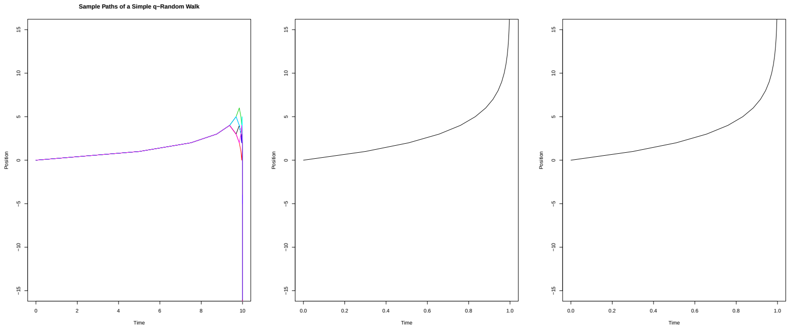


Figure 4: Generation of 5, More q-Random Walks with $\theta = 10$, $q = 0.5$, $n = 30$, $t = 10$.
Figure 5: Generation of 5 q-Random Walks Processes with $q = 0.7$, $n = 30$, $\theta = q^{-45}$, $t = 10$.
Figure 6: Generation of 5 More q-Random Walks Processes with $q = 0.7$, $n = 30$, $\theta = exp(30)$, $t = 10$.

A q-Brownian Motion

Theorem (Vamvakari(2017)

Consider the q-random Walk process with steps of length $\delta = c/\theta^{1/2}$, where $c = (q^{-1} - 1)/(\log q^{-1})$ is a normalization constant and $\theta = \theta_n = q^{-\alpha n}$, with $0 < \alpha < 1$. Then the q-random walk process is approximated, as $n \to \infty$, by a continuous analogue one $\{Y_q(t), t \geq 0\}$, where the r.v. $Y_q(t)$ is the position after time t with probability density function distribution

$$
\begin{aligned}
f(y,t) = & \frac{q^{-7/8}}{\sigma(2\pi)^{1/2}} \frac{\left(q^{-1}-1\right)^{1/2}}{(\log q^{-1})^{1/2}} \left(\frac{(1-q)^{1/2}}{q^{3/2}} \cdot \frac{(y-\mu_t)}{\sigma_t} + q^{-1} \right)^{-1/2} \\
& \cdot \exp\left(\frac{1}{2\log q} \log^2 \left(q^{-3/2}(1-q)^{1/2} \cdot \frac{(y-\mu_t)}{\sigma_t} + q^{-1} \right) \right), \\
& y > \mu_t - \sigma_t q^{1/2}(1-q)^{-1/2},
\end{aligned} \tag{1}
$$

where the mean value μ and the variance σ^2 of the r.v. $Y_q(t)$ are given by

$$
\mu_t = E(Y_q(t)) = ct, \;\; \sigma_t^2 = V(Y_q(t)) = \frac{1-q}{q}(ct)^2 + ct.
$$

q-Brownian Motion

Definition (Vamvakari (2017))

The continuous stochastic process $\{Y_q(t), t \geq 0\}$, is called *q-Brownian motion* with parameters q, μ_t and σ_t^2, if the following properties hold

(A) The distribution of the increment $Y_q(t_2) - Y_q(t_1)$, with $t_2 - t_1 = (1-q)t$, $t > 0$, is the linear transformed standardized Stieltjes-Wigert distribution with p.d.f (1), where $\mu_{t_2 - t_1} = c(1-q)t$ and $\sigma^2_{t_2-t_1} = \frac{1-q}{q}(c(1-q)t)^2 + c(1-q)t$.

(B) The increments $Y_q(t_k) - Y_q(t_{k-1})$, where $t_k - t_{k-1} = q^{k-1}(1-q)t$, $k = 1, 2, \ldots,$ are independent.

(C) $Y_q(0) \geq 0$ and $Y_q(t)$ is continuous at $t = 0$.

q-Brownian Motion

Theorem

Let $W_q(T) = \max_{0\leq t\leq T} Y_q(t)$ the r.v. of the maxima in the q-Brownian motion and T_b the first time passage of the process $Y_q(t)$ from the point b with $b > \mu_t - \sigma_t q^{1/2}(1-q)^{-1/2}$. Then

$$P\left(W_q(T) \geq b\right) = q^{-15/8}\left(1 - erf\left(B_t\right)\right)$$

and the p.d.f of the first time passage from b is given by

$$f_{T_b}(t) = -\frac{dB_t}{d_t}\frac{2q^{-15/8}}{\sqrt{\pi}} exp\left(-B_t^2\right),$$

where

$$B_t = \frac{1}{\sqrt{2logq^{-1}}}\left(log\left(q^{-3/2}(1-q)^{1/2}\frac{b-\mu_t}{\sigma_t} + q^{-1}\right) + \frac{logq}{2}\right).$$

q-Brownian Motions Simulations in R

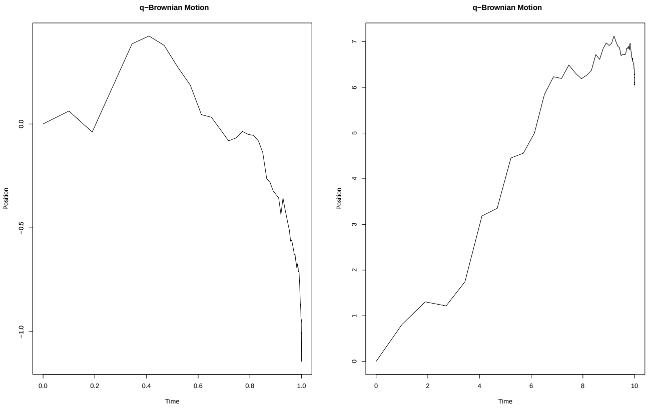


Figure 1. q-Brownian Motion Simulation in R with $q = 0.9$, $t = 10$
Figure 2. q-Brownian Motion Simulation in R with $q = 0.9$, $t = 1$

A q-Random Walk on the 2D Integer Lattice

A q-Random Walk on the 2D Integer Lattice

Definition

The Markov chain in which at each transition is likely to take one step to the right, left, up or down in the plane with varying transition probabilities given respectively by

$$P_{(i,j),(i+1,j)} = \frac{\theta_1 q^{i-1}}{(1+\theta_1 q^{i-1})(1+\theta_2 q^{j-1})},$$
$$P_{(i,j)(i-1,j)} = \frac{1}{(1+\theta_1 q^{i-1})(1+\theta_2 q^{j-1})},$$
$$P_{(i,j),(i,j+1)} = \frac{\theta_2 q^{j-1}}{(1+\theta_1 q^{i-1})(1+\theta_2 q^{j-1})},$$
$$P_{(i,j)(i,j-1)} = \frac{\theta_1 q^{i-1}\theta_2 q^{j-1}}{(1+\theta_1 q^{i-1})(1+\theta_2 q^{j-1})},\ i,j = 0, \pm 1, \pm 2, \ldots,$$

where $0 < \theta_1, \theta_2 < \infty$, $0 < q < 1$, is called q-random walk on the two-dimensional integer lattice.

A q-Random Walk on the 2D Integer Lattice

A q-Random Walk on the 2D Integer Lattice

Lemma

Let X^+ be the number of "right steps", X^- be the number of "left steps", Y^+ be the number of "up steps", and Y^- be the number of be "down steps", after $2n$ steps in the q-random walk on the 2D dimensional integer lattice, starting from the origin $\mathbf{0}$, with $\theta_1/\theta_2 = q^{n/2}$, $n = 0, 1, 2, \ldots$. Then, it holds that

$$P^n_{\mathbf{0},\mathbf{0}} = P\left(X^+ = x, X^- = x, Y^+ = n - x, Y^- = n - x\right) =$$

$$= \frac{\theta_2^{2n} q^{2\binom{n}{2}} \binom{2n}{x, x, n-x}_q q^{-x(n-x)} \left(\frac{\theta_1}{\theta_2}\right)^{2x}}{\prod_{i=1}^{2n} \left(1 + \theta_1 q^{i-1} + \theta_2 q^{n-i-1} + \theta_1\theta_2 q^{n-2}\right)}, x = 0, 1, .., n.$$

A q-Random Walk on the 2D Integer Lattice

Theorem

Let the q-random walk on the 2D dimensional integer lattice after $2n$ *steps, starting from the origin* $\mathbf{0}$, *with* $\theta_1/\theta_2 = q^{n/2}$, $n = 0, 1, 2, \ldots$. *Then for* θ_2 *constant or for* $\theta_2 = q^{-\alpha n}$, $0 < a < 1$, $n = 0, 1, 2, \ldots$, *the 2D q-random walk is recurrent, while for* $\theta_2 q^n \to \infty$, *as* $n \to \infty$, *is transient.*

A 2D q-Random Walk Stochastic Process

A 2D q-Random Walk Stochastic Process

- Let the time interval $(0, t]$, $t > 0$, and partition it into parts which are geometrically decreasing with rate q, defined by

$$\delta_i(n;t) = (n)_q^{-1} q^{i-1} t, \quad i = 1, 2, \ldots, n, \quad n \geq 1. \tag{2}$$

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- Let the process generated by taking one step, of length $\delta = 1$, to the right, left, up or down in the plane at every time period $(n)_q^{-1} q^{i-1} t$, $i = 1, 2, \ldots, n$ $n \geq 1$, with varying transition probabilities

$$\begin{aligned}
p_{i,1} &= P(X_i = \delta, Y_i = 0) = \frac{\theta_1 q^{i-1} t/(n)_q}{(1+\theta_1 q^{i-1} t/(n)_q)(1+\theta_2 q^{n-i-1} t/(n)_q)}, \\
p_{i,2} &= P(X_i = -\delta, Y_i = 0) = \frac{1}{(1+\theta_1 q^{i-1} t/(n)_q)(1+\theta_2 q^{n-i-1} t/(n)_q)}, \\
P_{i,3} &= P(X_i = 0, Y_i = \delta) = \frac{\theta_2 q^{n-i-1} t/(n)_q}{(1+\theta_1 q^{i-1} t/(n)_q)(1+\theta_2 q^{n-i-1} t/(n)_q)}, \\
P_{i,4} &= P(X_i = 0, Y_i = -\delta) = \frac{\theta_1 \theta_2 q^{n-2} t^2/(n)_q^2}{(1+\theta_1 q^{i-1} t/(n)_q)(1+\theta_2 q^{n-i-1} t/(n)_q)},
\end{aligned} \tag{3}$$

where $0 < \theta_1 < \infty, 0 < \theta_2 < \infty$, $0 < q < 1$ with $\theta_1/\theta_2 = q^{n/2}$. Then, at time $\sum_{i=1}^{n} \delta_i(n;t) = t$, the position of the process is the bivariate r.v. $(X_{n,q}(t), Y_{n,q}(t))$ with $X_{n,q}(t) = \sum_{i=1}^{n} X_i$ and $Y_{n,q}(t) = \sum_{i=1}^{n} Y_i$.

Definition

The continuous time bivariate stochastic process $\{(X_{n,q}(t), Y_{n,q}(t)), t > 0\}$, is called *bivariate q-random walk stochastic process* with parameters n, θ_1, θ_2 and q, $0 < q < 1$, if the following properties hold

(a) In each of the consecutive mutually disjoint time intervals of length $\delta_i(n;t) = (n)_q^{-1} q^{i-1} t$, $i = 1, 2, \ldots, n$, $n \geq 1$, at most one event (right or left or up or down step) occurs and

$$P\left(X_{n,q}\left(\frac{1-q^i}{1-q^n}t\right) - X_{n,q}\left(\frac{1-q^{i-1}}{1-q^n}t\right) = 1, Y_{n,q}\left(\frac{1-q^i}{1-q^n}t\right) - Y_{n,q}\left(\frac{1-q^{i-1}}{1-q^n}t\right) = 0\right)$$
$$= \frac{\theta_1 q^{i-1} t/(n)_q}{(1+\theta_1 q^{i-1} t/(n)_q)(1+\theta_2 q^{n-i-1} t/(n)_q)},$$
$$P\left(X_{n,q}\left(\frac{1-q^i}{1-q^n}t\right) - X_{n,q}\left(\frac{1-q^{i-1}}{1-q^n}t\right) = -1, Y_{n,q}\left(\frac{1-q^i}{1-q^n}t\right) - Y_{n,q}\left(\frac{1-q^{i-1}}{1-q^n}t\right) = 0\right)$$
$$= \frac{1}{(1+\theta_1 q^{i-1} t/(n)_q)(1+\theta_2 q^{n-i-1} t/(n)_q)},$$
$$P\left(X_{n,q}\left(\frac{1-q^i}{1-q^n}t\right) - X_{n,q}\left(\frac{1-q^{i-1}}{1-q^n}t\right) = 0, Y_{n,q}\left(\frac{1-q^i}{1-q^n}t\right) - Y_{n,q}\left(\frac{1-q^{i-1}}{1-q^n}t\right) = 1\right)$$
$$= \frac{\theta_2 q^{n-i-1} t/(n)_q}{(1+\theta_1 q^{i-1} t/(n)_q)(1+\theta_2 q^{n-i-1} t/(n)_q)},$$
$$P\left(X_{n,q}\left(\frac{1-q^i}{1-q^n}t\right) - X_{n,q}\left(\frac{1-q^{i-1}}{1-q^n}t\right) = 0, Y_{n,q}\left(\frac{1-q^i}{1-q^n}t\right) - Y_{n,q}\left(\frac{1-q^{i-1}}{1-q^n}t\right) = -1\right)$$
$$= \frac{\theta_1\theta_2 q^{n-2} t^2/(n)_q^2}{(1+\theta_1 q^{i-1} t/(n)_q)(1+\theta_2 q^{n-i-1} t/(n)_q)}, i = 1, 2, \ldots, n, \ 0 < \theta_1, \theta_2 < \infty, \ \theta_1/\theta_2 = q^{n/2}.$$

Definition cont.

(b) The bivariate increments $\left(X_{n,q}(t_i) - X_{n,q}(t_{i-1}), Y_{n,q}(t_i) - Y_{n,q}(t_{i-1})\right)$, where $t_i - t_{i-1} = (n)_q^{-1} q^{i-1} t$, $i = 1, 2, \ldots, n$, are independent.

(c) The process starts at time $t = 0$ with $\left(X_{n,q}(0), Y_{n,q}(0)\right) = (0, 0)$.

2D q-Random Walk Stochastic Processes Simulated in R

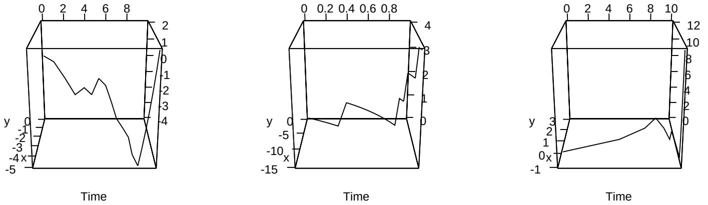

Figure 9: 2D q-Random Walk Process Simulation in R with $q = 0.9$, $n = 20$, $\theta_2 = q^{10}$, $t = 10$.

Figure 10: 2D q-Random Walk Process Simulation in R with $q = 0.9$, $n = 20$, $\theta_2 = q^{10}$, $t = 1$.

Figure 11: 2D q-Random Walk Process Simulation in R with $q = 0.5$, $n = 20$, $\theta_2 = q^{10}$, $t = 10$.

2D q-Random Walk Stochastic Processes Simulated in R

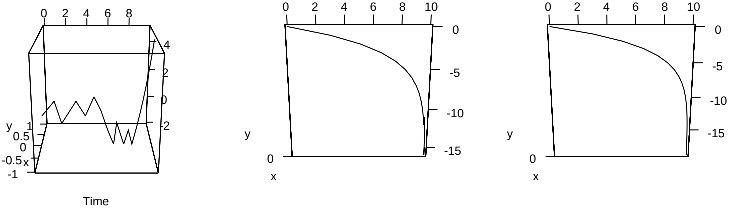

Figure 12: 2D q-Random Walk Process Simulation in R with $q = 0.7$, $n = 20$, $\theta_2 = q^{-10}$, $t = 10$.

Figure 13: 2D q-Random Walk Process Simulation in R with $q = 0.5$, $n = 20$, $\theta_2 = q^{-30}$, $t = 10$.

Figure 14: 2D q-Random Walk Process Simulation in R with $q = 0.5$, $n = 20$, $\theta_2 = e^{10}$, $t = 10$.

Future Plan

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- Embedment of q-random walks in random graphs

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Thank you!

R-Codes are available for anyone interested!