Combinatorial proofs for identities concerning Stirling number of the second kind for classical Coxeter groups

Eli Bagno
Jerusalem College of Technology, Israel

joint work with
Riccardo Biagioli (Lyon 1, France)
David Garber (Holon Institute of Technology, Israel)

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Stirling and Eulerian numbers

- The most common combinatorial interpretation of Stirling number of the second kind $S(n, k)$ is as counting the number of partitions of $[n] = \{1, \ldots, n\}$ into k blocks.
- The Eulerian number $A(n, k)$ counts the number of permutations in S_n, having $\mathrm{des}(\pi) = k - 1$, where $\mathrm{des}(\pi)$ is the size of the descent set of a permutation $\pi \in S_n$:

$$\mathrm{Des}(\pi) = \{i \in [n-1] \mid \pi(i) > \pi(i+1)\}$$

Two classical identities

Theorem (classical)

For all positive integers n and r, we have:

$$S(n,r) = \frac{1}{r!}\sum_{k=0}^{r} A(n,k)\binom{n-k}{r-k}$$

Theorem (classical)

Let $x \in \mathbb{R}$ and let $n \in \mathbb{N}$. Then:

$$x^n = \sum_{k=0}^{n} S(n,k)[x]_k$$

where $[x]_k = x(x-1)\cdots(x-k+1)$ is the **falling polynomial of degree k** *and $[x]_0 = 1$.*

- This is actually the original definition of Stirling numbers.
- A combinatorial proof, realizing x^n as the number of functions from the set $\{1,\ldots,n\}$ to the set $\{1,\ldots,x\}$, is presented in Stanley's EC1.

The Hyperoctahedral group B_n

Definition

A **signed permutation** is a permutation w on the set $\{\pm 1, \ldots, \pm n\}$ with the property that $w(-i) = -w(i)$ for all i.

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The **hyperoctahedral group** B_n is the group of signed permutations.

Definition

Consider an **order** on the set $\Sigma = \{\pm 1, \ldots, \pm n\}$:

$$-n < -(n-1) < \cdots < -1 < 1 < 2 < \cdots < n.$$

Flag descent of type B

Definition

For $\pi \in B_n$, define the parameter des_A:

$$\mathrm{des}_A(\pi) = |\{i \in [n] \mid \pi(i) > \pi(i+1)\}|; \qquad \varepsilon_1(\pi) = \begin{cases} 1 & \pi(1) < 0 \\ 0 & \pi(1) > 0. \end{cases}$$

The **flag descent** is defined as: $\mathrm{fdes}(\pi) = 2\mathrm{des}_A(\pi) + \varepsilon_1(\pi)$.

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Example

$$\pi = \begin{bmatrix} \bar{5} & \bar{4} & \bar{3} & \bar{2} & \bar{1} & & 1 & 2 & 3 & 4 & 5 \\ \bar{5} & 4 & \bar{1} & 2 & \bar{3} & & 3 & \bar{2} & 1 & \bar{4} & 5 \end{bmatrix}.$$

$$\mathrm{des}_A(\pi) = 2; \quad \varepsilon_1(\pi) = 0; \quad \mathrm{fdes}(\pi) = 2 \cdot 2 + 0 = 4$$

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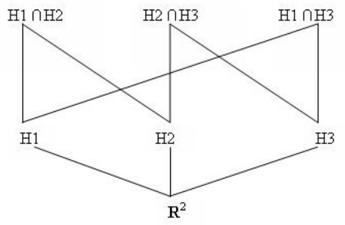
$$\mathrm{des}_A(\pi) = 2; \quad \varepsilon_1(\pi) = 0; \quad \mathrm{fdes}(\pi) = 2 \cdot 2 + 0 = 4$$

Definition

Denote by $A_B(n, k)$ the number of elements $\pi \in B_n$ satisfying $\mathrm{fdes}(\pi) = k - 1$.

Set partitions of type A

The lattice of set partitions $\pi_A(n)$ can be interpreted as the poset of intersection subspaces of subsets of hyperplanes in the Coxeter root system of type A_{n-1}: $\{\ x_i = x_j \mid 1 \le i < j \le n\ \}$ ordered by reverse inclusion.



Example

The partition $\{\{1,3,4\},\{2,5,6\}\}$ is interpreted as

$$\left\{\ \vec{x} = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6\ \middle|\ x_1 = x_3 = x_4,\ \ x_2 = x_5 = x_6\ \right\}$$

Set partitions of type B

The poset of intersections of subsets of the hyperplanes of the root system of type B:

$$\{\ x_i = \pm x_j \mid 1 \leq i < j \leq n\ \} \cup \{\ x_i = 0 \mid 1 \leq i \leq n\ \}$$

consists of subspaces which look typically like:

$$\{\ x_1 = -x_3 = x_6 = x_8 = x_9,\ \ x_5 = -x_7,\ \ x_2 = x_4 = 0\ \}$$

We can represent each intersection of subspaces in a simpler way like this:

$$\{\ \{1, -3, 6, 8, -9\}, \{-1, 3, -6, -8, 9\},\ \{5, -7\}, \{-5, 7\},\ \{2, -2, 4, -4\}\ \}.$$

Set partitions of type B

Definition

Definition: A **set partition of $[n]$ of type B** is a way to divide the set $\{\pm 1, \ldots, \pm n\}$ into blocks such that the following conditions are satisfied:

- If B appears as a block in the partition, then $-B$ (which is obtained by negating all the elements of B) also appears in that partition.
- There exists at most one block satisfying $-B = B$. This block is called the **zero block** (if it exists, it is a subset of $\{\pm 1, \ldots, \pm n\}$ of the form $\{\pm i \mid i \in C\}$ for some $C \subseteq [n]$).

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Definition

$S_B(n, r)$ denotes the number of set partitions of $\{1, \ldots, n\}$ of type B having exactly r blocks.

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Definition

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Example

$$\{\{2, -2, 4, -4\}, \ \{1, -3, 6, 8, -9\}, \{-5, 7\}, \{-1, 3, -6, -8, 9\}, \ \{5, -7\} \ \}.$$

Ordered set partition of type B

Definition

Definition: An **ordered set partition of $[n]$ of type B** is an ordered permutation of the elements of a set partition of $[n]$ of type B such that the following conditions are met:

- The zero-block, if exists, appears at the end.
- For each block B which in not a zero-block, the blocks B and $-B$ occupy adjacent places.

Example

$$\{\ \{5,-7\},\{-5,7\},\ \{1,-3,6,8,-9\},\{-1,3,-6,-8,9\},\ \{2,-2,4,-4\}\}.$$

Main Theorem 1 (for type B)

Theorem (B-Biagioli-Garber)

$$2^{\lfloor \frac{r}{2} \rfloor} \left\lfloor \frac{r}{2} \right\rfloor! \; S_B(n, r) = \sum_{k=1}^{r} A_B(n, k) \binom{n - \lceil \frac{k}{2} \rceil}{\lfloor \frac{r-k}{2} \rfloor}$$

Proof's idea:
LHS = Number of ordered set partitions of $[n]$ of type B.
RHS = Weighted sum of numbers of signed permutations classified by their flag descent.

Coxeter groups of type D

Definition

The group D_n is the subgroup of B_n consisting of all the signed permutations having an even number of negative entries

Definition

For $\pi \in D_n$, recall:

$$\mathrm{Des}_A(\pi) = \{\ i \ \mid\ \pi(i) > \pi(i+1), i \in \{1, \ldots, n-1\}\ \}.$$

The **descent set** $\mathrm{Des}^*_D(\pi)$ is defined as follows:

$$\mathrm{Des}^*_D(\pi) = \begin{cases} \mathrm{Des}_A(\pi) \cup \{0\} & \pi(1) + \pi(2) < 0 \\ \mathrm{Des}_A(\pi) & \text{otherwise} \end{cases}$$

$$\mathrm{des}^*_D(\pi) = |\mathrm{Des}^*_D(\pi)|$$

Example

Let $\pi = [-3, 2, 1] \in D_3$. Then:

$$\mathrm{Des}^*_D(\pi) = \{0, 2\}$$

since $(-3) + 2 < 0; \quad 2 > 1$.

Definition

Denote by $A^*_D(n, k)$ the number of elements $\pi \in D_n$ satisfying $\mathrm{des}^*_D(\pi) = k$.

Set partitions of type D

The poset of intersections of subsets of the hyperplanes of the root system of type D: $\{ x_i = \pm x_j \mid 1 \leq i < j \leq n \}$ consists of subspaces which look typically like:

$$\{ x_1 = -x_3 = x_6 = x_8 = x_9, \; x_5 = -x_7, \; x_2 = -x_4 = -x_2 = x_4 \}$$

$$= \{ x_1 = -x_3 = x_6 = x_8 = x_9, \; x_5 = -x_7, \; x_2 = x_4 = 0 \}$$

Definition

A **set partition of $[n]$ of type D** is a set partition of $[n]$ of type B with the additional restriction that the zero block, if presents, contains at least two pairs.

Example

The set partition of [3] of type B: $\{\{1,2\},\{-1,-2\},\{\pm 3\}\}$ is not a set partition of [3] of type D, while $\{\{1\},\{-1\},\{\pm 2,\pm 3\}\}$ is a set partition of [3] of type D.

Main Theorem 1 for type D

Definition

Denote by $S_D^*(n,r)$ the number of set partitions of $[n]$ of type D having exactly r pairs of non-zero blocks.

Definition

$$A_D^*(n,k) = |\{\pi \in D_n \mid |\mathrm{Des}_D^*(\pi)| = k\}|$$

Theorem (B-Biagioli-Garber)

$$2^r r! S_D^*(n,r) = n2^{n-1}(r-1)! S_{n-1,r-1} + \sum_{k=0}^{r} A_D^*(n,k)\binom{n-k}{r-k}$$

where $S_{n,r}$ is the usual Stirling number of the second kind.

Part II: Stirling numbers as transfer matrix for falling polynomials

Theorem (classic)

Let $x \in \mathbb{R}$ and let $n \in \mathbb{N}$. Then:

$$x^n = \sum_{k=0}^{n} S(n,k)[x]_k$$

where $[x]_k = x(x-1)\cdots(x-k+1)$ is the **falling polynomial of degree k** *and $[x]_0 = 1$.*

- This is actually the original definition of Stirling numbers.
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Falling polynomials of type B

Theorem

Let $x \in \mathbb{R}$ and let $n \in \mathbb{N}$. Then we have:

$$x^n = \sum_{k=0}^{n} S_B^*(n,k)[x]_k^B$$

where $[x]_k^B = (x-1)(x-3)\cdots(x-2k+1)$ *and* $[x]_0^B = 1$.

- **Bala**'s proof uses generating functions techniques.
- **Remmel and Wachs**'s combinatorial proof uses a model of $j-$non-attacking rooks.
- **Our first bijective proof** is based on the interpretation of x^n as the number of functions from the set $\{1, 2, \ldots, n\}$ to the set $\{1, 2, \ldots, x\}$

Lattice points in type B

Another bijective proof was proposed to us by Vic Reiner and is based on counting lattice points: $\{-m, \ldots, 1, 0, 1, \ldots, m\}^n$.

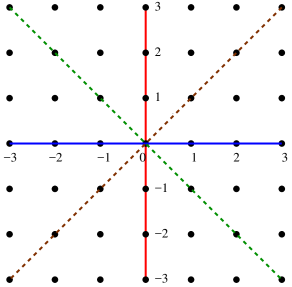

Figure: Lattice points for type *B*

- The L.H.S counts all the lattice points.
- The R.H.S counts the same thing according to the maximal intersection of hyperplanes of type *B* they lie on.

Falling polynomials of type D

Definition

Define the falling polynomial of type D:

$$[x]_k^D = (x-1)(x-3)\cdots(x-(2k-1))$$

for $1 \leq k < n$ and $[x]_n^D = (x-1)(x-3)\cdots(x-(2n-3))(x-(n-1))$ and $[x]_0^D = 1$

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Theorem (B-Biagioli-Garber)

For each $n \in \mathbb{N}$:

$$x^n = n([x]_{n-1}^D - (x-1)^{n-1}) + \sum_{k=0}^{n} S_D^*(n,k)[x]_k^D.$$

The proof: Lattice points in type D

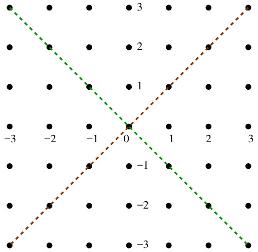

Figure: Lattice points for type D

- The L.H.S counts all the lattice points.
- The R.H.S counts the same thing according to the maximal intersection of hyperplanes of type D they lie on.
- The missing part: $n([x]_{n-1}^{D} - (x-1)^{n-1})$ counts the points which contain exactly one appearance of 0 and at least two non-zero coordinates are assigned the same absolute value.

Back to Theorem 1 for type B: Proof's idea

Recall:

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From a signed permutation to an ordered set partition of type B:

Let $\pi \in B_n$ with $\mathrm{fdes}(\pi) = k - 1$ be written in complete notation:

$$\pi = \begin{bmatrix} \bar{5} & \bar{4} & \bar{3} & \bar{2} & \bar{1} & & 1 & 2 & 3 & 4 & 5 \\ \bar{5} & 4 & \bar{1} & 2 & \bar{3} & & 3 & \bar{2} & 1 & \bar{4} & 5 \end{bmatrix}.$$

Divide the negative part into blocks by putting separators in any place which is a descent and reflect these separators to the positive part.

In our example:

$$\pi = [\ \bar{5} \quad 4 \mid \bar{1} \quad 2 \mid \bar{3} \qquad 3 \mid \bar{2} \quad 1 \mid \bar{4} \quad 5\ ].$$

Perform the following two steps:

1. If exists, move the block B containing the set $\{\pi(-1), \pi(1)\}$ to the end (*zero block*).
2. For each block B contained in the negative part of π, locate the block $-B$ right after it.

Example

$$\pi = \left[\begin{array}{ccccccccccc} \bar{5} & \bar{4} & \bar{3} & \bar{2} & \bar{1} & & 1 & 2 & 3 & 4 & 5 \\ \bar{5} & 4 & \bar{1} & 2 & \bar{3} & & 3 & \bar{2} & 1 & \bar{4} & 5 \end{array} \right].$$

Write the second line with the added separators:

$$\pi = [\ \bar{5} \quad 4 \mid \bar{1} \quad 2 \mid \bar{3} \qquad 3 \mid \bar{2} \quad 1 \mid \bar{4} \quad 5\ ].$$

The associated ordered set partition of type B of π is:

$$(\{\bar{5}, 4\}, \{\bar{4}, 5\}, \{\bar{1}, 2\}, \{\bar{2}, 1\}, \{\bar{3}, 3\}).$$

If $r = k$, then we have associated to the signed permutation π an ordered set partition of type B and we are done.

If $r > k$, refine the partition by simultaneously splitting pairs of blocks of the form B and $-B$ (where $B \neq -B$), or by splitting a zero-block.

Example

$$\pi = \left[\begin{array}{ccccc|ccccc} \bar{5} & \bar{4} & \bar{3} & \bar{2} & \bar{1} & 1 & 2 & 3 & 4 & 5 \\ \bar{5} & 4 & \bar{1} & 2 & \bar{3} & 3 & \bar{2} & 1 & \bar{4} & 5 \end{array} \right].$$

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Refinement:

$$\pi = \left[\ \bar{5} \,\vdots\, \ 4 \ \middle|\ \bar{1} \quad 2 \ \middle|\ \bar{3} \qquad 3 \ \middle|\ \bar{2} \quad 1 \ \middle|\ \bar{4} \,\vdots\, \ 5\ \right].$$

The associated ordered set partition of type B of π is:

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Proof's idea of Theorem 1 for type D

Theorem (B-Biagioli-Garber)

$$2^r r! S_D^*(n,r) = n2^{n-1}(r-1)! S_{n-1,r-1} + \sum_{k=0}^{r} A_D^*(n,k) \binom{n-k}{r-k}$$

where $S_{n,r}$ is the usual Stirling number of the second kind.

Proof's idea:

Given $\pi \in D_n$, put a separator in places of descents.

In each place that there is no descent, we decide whether to insert an artificial separator or not.

Example

Example

Let $\pi = [3, 7, -1, 2, 4, -6, -5] \in D_7$. After placing the separators induced by the descents:

$$\pi = [3, 7 \mid -1, 2, 4 \mid -6, -5].$$

We add one artificial separator in the following way:

$$\pi = [3, 7 \mid -1 \,\vdots\, 2, 4 \mid -6, -5].$$

The obtained set partition is:

$$\{\{\pm 3, \pm 7\}, \{-1\}, \{1\}, \{2, 4\}, \{-2, -4\}, \{-5, -6\}, \{5, 6\}\}.$$

The problem: We obtain a set partition of type B which is not a set partition of type D - a zero block having exactly one pair.
When does this happen? If after placing all separators, there exists a separator between $\pi(1)$ and $\pi(2)$, but not before $\pi(1)$ (as $\pi(1) + \pi(2) > 0$).
Solution: Toggle the sign of $\pi(1)$, and then locate the separator which was been between $\pi(1)$ and $\pi(2)$ before $\pi(1)$, emulating a descent in position 0 (called **switch operation**); yielding a permutation $\pi' \in B_n - D_n$, whose blocks form a set partition of type D.

Example

Let $\pi = [3, -1, 4, -2, -6, -5] \in D_6$. After placing the separators:

$$\pi = [3 \mid -1, 4 \mid 2 \mid -6, -5].$$

$3 + (-1) > 0 \;\Rightarrow\; 0 \notin \mathrm{Des}_D(\pi)$,
so we get a set partition of type B which is not of type D:

$$\{\{\pm 3\}, \{-1, 4\}, \{1, -4\}, \{-5, -6\}, \{5, 6\}\}.$$

Switch operation:

$$\pi' = [-3, -1, 4, -2, -6, -5] \in B_6 - D_6.$$

After placing the separators induced by the descents, we have:

$$\pi' = [\; \mid -3, -1, 4 \mid -2 \mid -6, -5] \in B_6 - D_6,$$

which induces the set partition of type D:

$$\{\{-3, -1, 4\}, \{3, 1, -4\}, \{-2\}, \{2\}, \{-6, -5\}, \{6, 5\}\}.$$

Definition

Definition: An **odd partition** is a partition of type D contains an odd number of negative numbers (induced by $\pi' \in B_n - D_n$).

Lemma (Structure of missing odd partitions)

The ordered odd partitions of type D having r blocks which cannot be obtained from permutations in D_n by a switch operation are exactly of the form

$$P' = \{\{*\}, P\},$$

where $*$ stands for one element from $\{\pm 1, \ldots, \pm n\}$ and P is the blocks of a usual ordered set partition of the set $[n] - \{*\}$ having $r-1$ blocks, such that an odd number of the elements of $\{1, \ldots, n\}$ are signed in P'.

Lemma (Counting missing odd partitions)

The number of odd partitions which cannot be obtained from permutations in D_n by a switch operation is:

$$n2^{n-1}(r-1)!S_{n-1,r-1}.$$

Thank you!