On the Largest Part Size and Its Multiplicity of a Random Integer Partition

GASCom 2018, Athens, Greece

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Integer partitions

If n is a positive integer, then, by *a partition* λ of n, we mean the representation

$$\lambda : \quad n = \lambda_1 + \lambda_2 + . + \lambda_k, \quad k \geq 1, \tag{1}$$

where the positive integers λ_j satisfy $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k$. Let $\Lambda(n)$ be the set of all partitions of n and let $p(n) = |\Lambda(n)|$ (by definition $p(0) = 1$ regarding that the one partition of 0 is the empty partition). Euler has proved the following well known and beautiful generating function identity for the numbers $\{p(n)\}$:

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{j=1}^{\infty} \frac{1}{1 - x^j} := P(x). \tag{2}$$

The number $p(n)$ is determined asymptotically by the famous partition formula of Hardy and Ramanujan (1918):

$$p(n) \sim \frac{c_0}{n} e^{c_1\sqrt{n}}, c_0 = \frac{1}{4\sqrt{3}}, c_1 = \pi\sqrt{\frac{2}{3}}, \quad n \to \infty.$$

(The proof is based on the Euler's infinite product identity (2) and the circle method.) A precise asymptotic expansion for $p(n)$ was found by Rademacher (1937) (for more details, see also Andrews (1976); Chapter 5).

Ferrers diagrams; an example

Each partition $\lambda \in \Lambda(n)$ has a unique graphical representation called Ferrers diagram (see, e.g., Andrews (1976); Chaapter 1). It illustrates partition (1) by the two-dimensional array of dots, composed by λ_1 dots in the first (most left) row, λ_2 dots in the second row,..., and so on, λ_k dots in the kth row. Therefore, a Ferrers diagram may be considered as a union of disjoint blocks (rectangles) of dots whose side lengths represent the part sizes and their multiplicities of the partition λ, respectively.

For instance Figure 1 illustrates the partition $7+5+5+5+4+2+1+1+1$ of $n=31$:

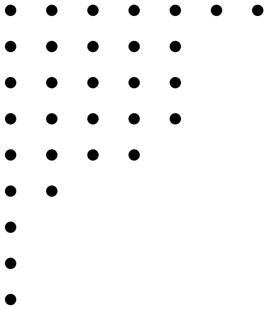

Figure 1

Integer partition statistics

Further on, we assume that, for fixed integer $n \geq 1$, a partition $\lambda \in \Lambda(n)$ is selected uniformly at random. In other words, we assign the probability $1/p(n)$ to each $\lambda \in \Lambda(n)$. We denote this probability measure on $\Lambda(n)$ by $\mathbb{P}$. Let $\mathbb{E}$ be the expectation with respect to $\mathbb{P}$. In this way, each numerical characteristic of $\lambda \in \Lambda(n)$ can be regarded as a random variable, or, a statistic produced by the random generation of partitions of n.

In this work, we focus on two statistics of random integer partitions $\lambda \in \Lambda(n)$: $L_n = L_n(\lambda) = \lambda_1$, which is the largest part size in representation (1) and $M_n = M_n(\lambda)$, equal to the multiplicity of the largest part λ_1 (i.e., $M_n(\lambda) = m, 1 \leq m \leq k-1$, if $\lambda_1 = ... = \lambda_m > \lambda_{m+1} \geq ... \geq \lambda_k$ in (1), and $M_n(\lambda) = k$, if $\lambda_1 = ... = \lambda_k$). In particular, in Figure 1 $L_{31} = 7, M_{31} = 1$.

The largest part size L_n

The earliest asymptotic results on random integer partition statistics have been obtained long ago by Husimi (1938) and Erdös and Lehner (1941).
Husimi has derived an asymptotic expansion for $\mathbb{E}(L_n)$ in the context of a statistical physics model of a Bose gas.
Erdös and Lehner were apparently the first who have studied random partition statistics in terms of probabilistic limit theorems.
In fact, they showed that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{L_n}{\sqrt{n}} - \frac{1}{2c}\log n \le u\right) = H(u), \tag{3}$$

where

$$H(u) = \exp\left(-\frac{1}{c}e^{-cu}\right), \quad c = \frac{\pi}{\sqrt{6}}, \quad -\infty < u < \infty. \tag{4}$$

Later on, Szekeres (1953) has studied in detail the asymptotic behavior of the number of integer partitions of n whose largest part is $\leq k$ and $= k$ in the whole range of values of $k = k(n)$. In particular, he has obtained Erdös and Lehner's limit theorem (3) using an entirely different method of proof.
Husimi's asymptotic result was subsequently reconfirmed by Kessler and Livingston (1976). Higher moments of L_n were studied by Richmond (1974). A general method providing asymptotic expansions of expectations of integer partition statistics was recently proposed by Grabner, Knopfmacher and Wagner (2014).

Among the numerous examples, they derived a rather complete asymptotic expansion for $\mathbb{E}(L_n)$, namely,

$$\mathbb{E}(L_n) = \frac{\sqrt{n}}{2c}(\log n + 2\gamma - 2\log c) + \frac{\log n}{2c^2} + \frac{1}{4} + \frac{1 + 2\gamma - 2\log c}{4c^2} + O\left(\frac{\log n}{n}\right), \quad n \to \infty, \tag{5}$$

where $c = \pi/\sqrt{6}$ and $\gamma = 0.5772...$ denotes the Euler's constant (see Grabner et al. (2014); Proposition 4.2).
We also notice that by conjunction of the Ferrers diagram the largest part and the total number of parts in a random partition of n are identically distributed for any n. The sequence $\{p(n)\mathbb{E}(L_n)\}_{n\geq 1}$ is given in the On-line Encyclopedia of Integer Sequences by N. Sloane as A006128.

The multiplicity M_n of the largest part

There are serious reasons to believe that the multiplicity M_n of the largest part of a random partition of n behaves asymptotically in a much simpler way than many other partition statistics.
Grabner and Knopfmacher (2006) used Erdös-Lehner limit theorem (3) to establish that

$$\lim_{n\to\infty} \mathbb{E}(M_n) = 1. \tag{6}$$

In addition, Fristedt (1993) showed that, with probability tending to 1 as $n \to \infty$, the first m_n largest parts in a random partition of n are distinct if $m_n = o(n^{1/4})$. Hence it may not be that L_n constitutes the main contribution to n by a single part size and some smaller part sizes may occur with sufficient multiplicity so that the products of these part sizes with their multiplicities could be much larger than L_n. In terms of the Ferrers diagram this means that its first block has typically smaller area than several next block areas with larger heights (multiplicities of parts).

The aim of this study

Our aim in this work is to study the asymptotic behavior of the area $L_n M_n$ of the first block in the Ferrers diagram of a random partition of n. We show some similarities and differences between the single part size L_n and its corresponding block area $L_n M_n$.

Main results

As a first step, we obtain a distributional result for M_n that confirms the limit of $\mathbb{E}(M_n)$ in (6).

Theorem 1. *For any $n \geq 1$, we have*

$$\mathbb{P}(M_n = 1) = \frac{p(n-1)}{p(n)}.$$

In addition, if $n \to \infty$, then

$$\mathbb{P}(M_n = 1) = 1 - \frac{c}{\sqrt{n}} + \frac{1 + c^2/2}{n} + O(n^{-3/2}),$$

where $c = \pi/\sqrt{6}$.

Combining the Erdös-Lehner limit theorem (3) with Theorem 1 (ii), one can easily observe that the limiting distributions of L_n and $L_n M_n$ coincide under the same normalization.

Corollary. *For any real u, we have*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{L_n M_n}{\sqrt{n}} - \frac{1}{2c}\log n \le u\right) = H(u),$$

where $H(u)$ is the Gumbel's distribution function given by (4) and c is the same as in Theorem 1.

Although L_n and $L_n M_n$ follow the same limiting distribution, the difference in means $\mathbb{E}(L_n M_n) - \mathbb{E}(L_n)$ grows as fast as $\frac{1}{2}\log n$. A complete estimate is given in the following

Theorem 2. *If $n \to \infty$, then, as $n \to \infty$,*

$$\mathbb{E}(L_n M_n) = \mathbb{E}(L_n) + \frac{1}{2}\log n - C + O(1/\log n),$$

where $C = \log c + 1 - \gamma = 0.67165...$ and $\mathbb{E}(L_n)$ and c is the same as in Theorem 1.

Remark 1. The sequence $\{p(n)\mathbb{E}(L_n M_n)\}_{n\geq 1}$ is given as A092321 in the On-line Encyclopedia of Integer Sequences by N. Sloane.

The method of proof

The proofs of Theorems 1 and 2 are based on generating function identities established Archibald, Blecher, Brennan, Knopfmacher and Mansour (2016) that involve products of the form $P(x)G(x)$, where $P(x)$ is the Euler partition generating function (2) and $G(x)$ is a function which is analytic in the open unit disk and does not grow too fast as $x \to 1$. Our asymptotic expansions in Theorems 1 and 2 are obtained using a general asymptotic result of Grabner, Knopfmacher and Wagner (2014) for the nth coefficient $x^n[P(x)G(x)]$. For the sake of completeness, we summarize these results in the following two lemmas. The first one presents the generating function identities that we need; the second one is the base for our asymptotic analysis.

Lemma 1 (Archibald et al. (2016)).
(i) For any positive integer m, we have

$$\sum_{n=1}^{\infty} p(n)\mathbb{P}(M_n = m)x^n = x^m \prod_{j \geq m} (1 - x^j)^{-1} = P(x)x^m \prod_{j=1}^{m-1} (1 - x^j).$$

(ii) We have

$$\sum_{n=1}^{\infty} p(n)\mathbb{E}(L_n M_n)x^n = P(x)(F_1(x) + F_2(x)),$$

where

$$F_1(x) = \sum_{k=1}^{\infty} kx^k \prod_{j=k+1}^{\infty} (1 - x^j), \tag{7}$$

and

$$F_2(x) = \sum_{k=1}^{\infty} \frac{kx^{2k}}{1 - x^k} \prod_{j=k+1}^{\infty} (1 - x^j). \tag{8}$$

The next lemma presents two particular cases of a general result due to Grabner et al. (2014); Theorem 2.3.

Lemma 2. *Suppose that, for some constants $K > 0$ and $\eta < 1$, the function $G(x)$ satisfies*

$$G(x) = O(e^{K/(1-|x|)^{\eta}}), \quad |x| \to 1. \tag{9}$$

(i) Let $G(e^{-t}) = at^b + O(|f(t)|)$ as $t \to 0$, $\Re t > 0$, where $b \geq 0$ is an integer and a is real number. Then, as $n \to \infty$, we have

$$\frac{1}{p(n)} x^n [P(x)G(x)] = a \left(\frac{2\pi}{\sqrt{24n-1}} \right)^b \frac{s}{s-1}$$
$$\times \sum_{j=0}^{b+1} \frac{(b+j+1)!}{j!(b+j-1)!} \left(-\frac{1}{2s} \right)^j$$
$$+ O(e^{-2s}) + O\left(e^{-n^{1/2-\epsilon}} + f(c/\sqrt{n} + O(n^{-1/2-\epsilon})) \right)$$

for any $\epsilon \in (0, (1-\eta)/2)$, *where*

$$s = \sqrt{\frac{2\pi^2}{s}\left(n - \frac{1}{24}\right)} = 2c\sqrt{n - \frac{1}{24}}, \quad c = \frac{\pi}{\sqrt{6}}.$$

(ii) Suppose that $G(x)$ *satisfies condition (9) of part (i) and, for* $t = u + iv$, *let* $G(e^{-t}) = a \log \frac{1}{t} + O(f(\mid t \mid))$ *as* $t \to 0$, *where* $u > 0$, $v = O(u^{1+\epsilon})$ *as* $u \to 0^+$, *where* ϵ *and* a *are as in part (i). Then, as* $n \to \infty$, *we have*

$$\frac{1}{p(n)} x^n [P(x)G(x)] = a \log\left(\frac{\sqrt{24n-1}}{2\pi}\right) + O\left(n^{-1/2} + f\left(c/\sqrt{n} + O(n^{-1/2-\epsilon})\right)\right)$$

with c *as in part (i).*

Important role in the proof of Theorem 2 play the asymptotic expansions of $F_j(e^{-t}), j = 1, 2$ as $t \to 0$. Grabner et al. (2014) have studied $F_1(e^{-t})$ using Mellin transform technique. For $F_2(e^{-t})$ we are able to prove the following lemma using a classical summation method of a power series around its main singularity.

Lemma 3. *If $t = u + iv$ and u and v satisfy the conditions of Lemma 2(ii), then*

$$F_2(e^{-t}) = \log\frac{1}{t} + \gamma - 1 + O\left(1/\log\frac{1}{t}\right), \quad t \to 0.$$

Concluding remarks

Erdös and Lehner's limit theorem (3) and Theorem 1 show that the weak convergence of both partition statistics L_n and L_nM_n is controlled by the leading term in the asymptotic expansions of their expectations, equal to $\frac{\sqrt{n}}{2c}\log n$. Both statistcs, after one and the same normalization, tend to a Gumbel distributed random variable. The expectations of L_n and L_nM_n are, however, different for large n. In fact, by Theorem 2

$$\lim_{n\to\infty}\left(\mathbb{E}(L_nM_n) - \mathbb{E}(L_n) - \frac{1}{2}\log n\right) = -C = -0.67165....$$

This phenomenon suggests a question related to the typical shape of a random Ferrers diagram of n. To state the problem in a more precise way, we denote by $X_n^{(k)}$ the multiplicity of part k $(k = 1, ..., n)$ in a random integer partition of n. Let $Z_n^{(r)}$ be the rth largest member of the sequence $\{kX_n^{(k)}\}_{k=1}^n$.

Erdös and Szalay (1984) showed that $Z_n^{(1)}$, appropriately normalized, tends to a Gumbel distributed random variable. Fristedt (1993); Theorem 2.7 generalized this result to $Z_n^{(r)}$, for $r \geq 1$ fixed, and showed that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{c}{\sqrt{n}} Z_n^{(r)} - \frac{1}{2}\log\frac{n}{c^2} - \log\log\log n \leq u\right) = \int_{-\infty}^{u} \frac{\exp\left(-e^{-w} - rw\right)}{(r-1)!} dw, \quad -\infty < u < \infty. \tag{10}$$

So, it might be interesting to determine the typical position of $L_n M_n$ among all ordered areas $Z_n^{(r)}$.

More precisely, if R_n denotes the smallest value of r such that $Z_n^{(r)} = L_n M_n$, then we conjecture that

$$\mathbb{E}(R_n) \asymp \log\log n, \quad n \to \infty.$$

This claim is supported by a calculation of the expectation of the distribution in the right-hand side of (10) demonstrated by Fristedt (1993); p. 708.

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