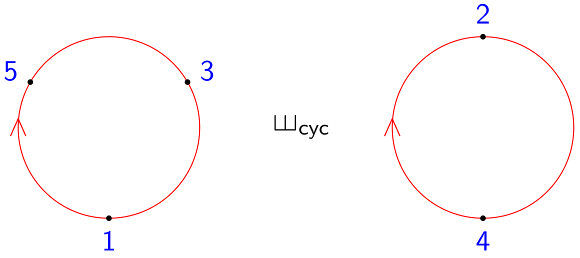
On Cyclic Descents

Ron Adin (Bar-Ilan U), Sergi Elizalde (Dartmouth),
Victor Reiner (U Minnesota), Yuval Roichman (Bar-Ilan U)



GASCom '18, Athens

Descents and cyclic descents of permutations

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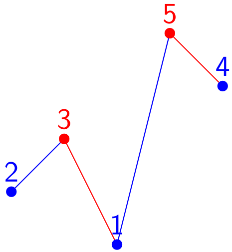

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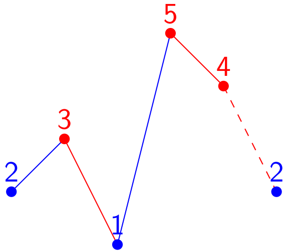

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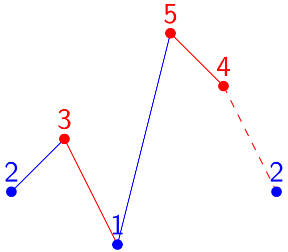

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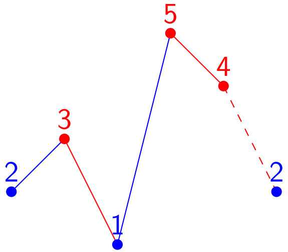

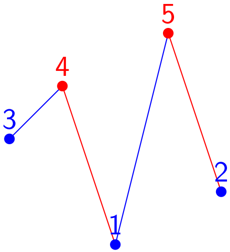

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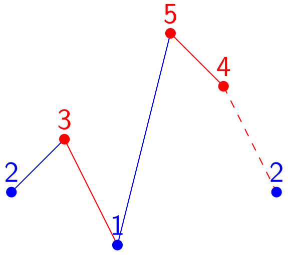

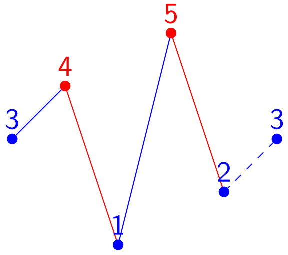

Cyclic Shuffles

A shuffle of two permutations with disjoint support $\sigma \in \mathfrak{S}_A$ and $\tau \in \mathfrak{S}_B$ is the set of all permutations in $\mathfrak{S}_{A \sqcup B}$, whose restriction to the letters in A is σ and restriction to the letters in B is τ.

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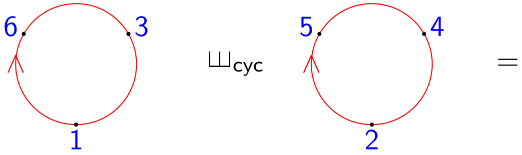

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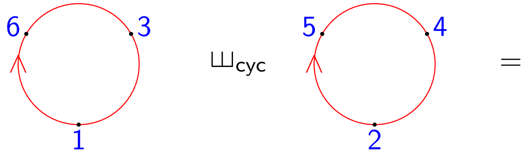

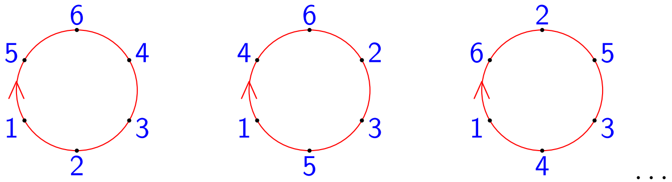

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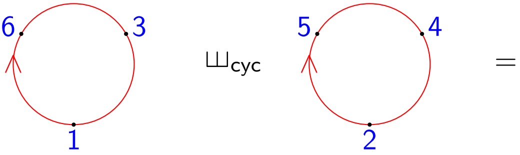

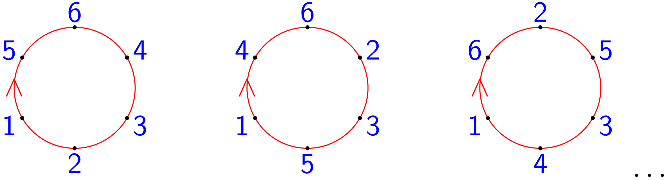

Problem 1:

What is the distribution of cyclic descents over cyclic shuffles ?

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Problem 2:

Define a cyclic descent set for SYT of any shape λ.

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Examples

- $\mathcal{T} = \mathfrak{S}_n$, with Cellini's cyclic descent set and $\mathbb{Z}_n$-action by cyclic rotation.
- $\mathcal{T} = \mathrm{SYT}(r^{n/r})$, with Rhoades' cyclic descent set and $\mathbb{Z}_n$-action by promotion.

A non-Escher property

"Ascending and Descending", M. C. Escher

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The paradox of $\mathrm{cDes}(\pi) = \varnothing$ and $\mathrm{cDes}(\pi) = [n]$.

Special shapes

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Each of the following shapes carries a cyclic descent extension:

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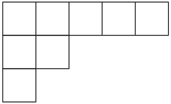

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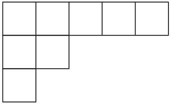

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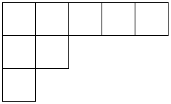

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The proofs are explicit and combinatorial.

Special shapes

Explicit definitions of cDes maps

Theorem (AER '16)

The shape $(n-k, 2, 1^{k-2})$ *has a unique cyclic descent map. For* $T \in \mathrm{SYT}(n-k, 2, 1^{k-2})$*, let* $n \in \mathrm{cDes}(T)$ *if and only if* $T_{2,2} - 1$ *appears in the first column of* T*.*

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Example The only cyclic descent map for $\lambda = (3, 2)$ is given by

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}.$$

$$\{1, 4\} \qquad \{2, 5\} \qquad \{3, 1\} \qquad \{4, 2\} \qquad \{5, 3\}$$

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There exists a cyclic descent map for the shape $(n-k,k)$*. For* $T \in \mathrm{SYT}(n-k,k)$*, let* $n \in \mathrm{cDes}(T)$ *if and only if both conditions hold:*

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Corollary (AER '17)

For any $2 \leq k \leq n/2$

$$\sum_{T \in \mathrm{SYT}(n-k,k)} q^{\mathrm{cdes}(T)} = \sum_{d=1}^{k} \frac{n}{d} \left[\binom{k-1}{d-1} \binom{n-k-1}{d-1} - \binom{k-2}{d-1} \binom{n-k}{d-1} \right] q^d .$$

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2. *If* λ *is not a hook then for all cyclic descent extensions, the* *distribution* *of* cDes *over* $\mathrm{SYT}(\lambda)$ *is* *uniquely* *determined.*

Quasi-symmetric functions

Definition A quasi-symmetric function is a formal power series $f \in \mathbb{Z}[[x_1, x_2, \ldots]]$ such that, for any $t \geq 1$, any two increasing sequences $i_1 < \ldots < i_t$ and $j_1 < \ldots < j_t$ of positive integers, any sequence $m = (m_1, \ldots, m_t)$ of positive integers, the coefficients of $x_{i_1}^{m_1} \cdots x_{i_t}^{m_t}$ and $x_{j_1}^{m_1} \cdots x_{j_t}^{m_t}$ in f are equal.

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Observation

$$Sym_n \subseteq QSym_n.$$

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Theorem (Gessel '84)

1. $\{F_{n,J} : J \subseteq [n-1]\}$ *is a basis of* $QSym_n$.

Quasi-symmetric functions

Recall the Schur function basis for Sym_n, $\{s_\lambda : \lambda \vdash n\}$.

Definition Given a subset $J \subseteq [n]$ define the fundamental quasi-symmetric function

$$F_{n,J} := \sum_{\substack{w_i \le w_{i+1} \\ w_i < w_{i+1} \ \forall i \in J}} x_{w_1} x_{w_2} \cdots x_{w_n}.$$

Theorem (Gessel '84)

1. $\{F_{n,J} : J \subseteq [n-1]\}$ *is a basis of* $QSym_n$.
2. *For every partition* $\lambda \vdash n$

$$\text{coeff}_{F_{n,J}}(s_\lambda) = \#\{T \in \text{SYT}(\lambda) : \text{Des}(T) = J\}.$$

Cyclic quasi-symmetric functions

Definition A *cyclic quasi-symmetric function* is $f \in \mathbb{Z}[[x_1, x_2, \ldots]]$
s.t. for any $t \geq 1$, $i_1 < \ldots < i_t$ and $j_1 < \ldots < j_t$,
$m = (m_1, \ldots, m_t) \in \mathbb{N}^t$ and $d \in \mathbb{Z}_t$,
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Problem 3:

Find a cyclic analogue of $F_{n,J}$, which satisfies

$$\text{coeff}_{F_{n,J}^{\text{cyc}}}(s_\lambda) \geq 0 \qquad (\forall \lambda \vdash n, J \subseteq [n]).$$

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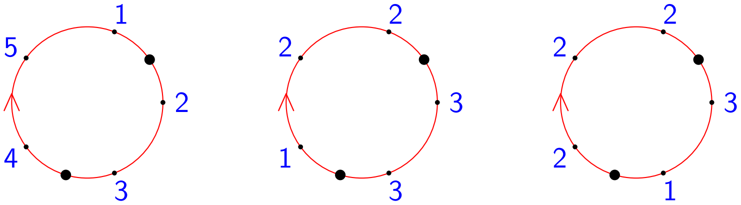
Example

Let $n = 5$ and $J = \{1, 3\}$.



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Prop. $\{F^{\text{cyc}}_{n,J} : \varnothing \subsetneq J \subseteq [n]\}$, where J runs over representatives of the $\mathbb{Z}_n$-action on $2^{[n]}$ by cyclic shifts, is a basis of $cQSym_n$.

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Theorem (Adin-Gessel-Reiner-R '18)

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Proof Idea: Apply the existence of a map $\mathrm{cDes} : \mathrm{SYT}(\lambda) \longrightarrow 2^{[n]}$, which statisfies cyclic descent extension axioms, to show that

$$\frac{1}{n} \sum_{T \in \mathrm{SYT}(\lambda/\mu)} F^{\mathrm{cyc}}_{n,\mathrm{cDes}(T)} = s_{\lambda/\mu}.$$

Structure constants and equi-distribution

Structure Constants Theorem (AGRR '18)

Given a set partition $[n+m] = A \sqcup B$ *with* $|A| = m$ *and* $|B| = n$, $\sigma \in \mathfrak{S}_A$ *and* $\tau \in \mathfrak{S}_B$*, the following expansion holds*

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Corollary (AGRR '18)

Given two permutations σ and τ of disjoint sets of integers, the distribution of the cyclic descent set over all cyclic shuffles of $[\sigma]$ and $[\tau]$ depends only on $\text{cDes}([\sigma])$ and $\text{cDes}([\tau])$.

Distribution of cyclic descents over cyclic shuffles

Theorem (AGRR '18)

Given a set partition $[n+m] = A \sqcup B$ with $|A| = m$ and $|B| = n$, $\sigma \in \mathfrak{S}_A$ and $\tau \in \mathfrak{S}_B$, the number of cyclic shuffles of $[\sigma]$ and $[\tau]$ with cyclic descent number k is equal to

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where $i = \mathrm{cdes}(\sigma)$ and $j = \mathrm{cdes}(\tau)$.

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where $\mathbb{Z}[[q]]_\odot$ is defined by equipping $\mathbb{Z}[[q]]$ with the product

$$q^i \odot q^j := q^{\max(i,j)}.$$

Thank You and

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