Permutation statistics for a percolation model on $\mathbb{Z}^2$ with imposed symmetries

Henri Derycke
PhD Advisor: Yvan Le Borgne

LaBRI, Université de Bordeaux

GASCom 2018, June 18-20, Athens, Greece

Simplified sandpile model on grid graph

Froböse local model: Bootstrap percolation [GH08]

Simplification: symmetry on the sampling of $C(i,j)$

Probabilities described by permutation statistics

From rectangular growth to partial permutations

Counter automaton for a canonical growth

Counter automaton for partial permutations

Automata identification

Decreasing subsequences and q-positivity

$\sum_{\sigma \in ??} \frac{q^{\mathrm{maj}\,\sigma} - q^{\mathrm{inv}\,\sigma}}{1-q} \in \mathbb{N}[q]$

Proof via Kadell weighted inversion number

Let $\mathcal{C} = (c_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \{0,1\}^{\mathbb{Z}^2}$ be a **configuration** of the square lattice $\mathbb{Z}^2$.

0	0	0	1	0	1	0	0	0
0	0	0	0	0	0	0	1	1
0	0	0	0	1	0	0	0	1
0	0	0	1	0	1	0	0	0
0	1	0	1	1	0	1	0	0
0	1	1	1	0	1	0	0	0
0	0	0	0	0	0	0	1	0
1	0	0	0	0	0	0	0	0
0	1	0	0	1	1	0	0	1

Let $\mathcal{C} = (c_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \{0,1\}^{\mathbb{Z}^2}$ be a **configuration** of the square lattice $\mathbb{Z}^2$.

0	0	0	1	0	1	0	0	0
0	0	0	0	0	0	0	1	1
0	0	0	0	1	0	0	0	1
0	0	0	1	0	1	0	0	0
0	1	0	1	1	0	1	0	0
0	1	1	1	0	1	0	0	0
0	0	0	0	0	0	0	1	0
1	0	0	0	0	0	0	0	0
0	1	0	0	1	1	0	0	1

- rule 0: the origin $(0,0)$ is active

Let $\mathcal{C} = (c_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \{0,1\}^{\mathbb{Z}^2}$ be a **configuration** of the square lattice $\mathbb{Z}^2$.

0	0	0	1	0	1	0	0	0
0	0	0	0	0	0	0	1	1
0	0	0	0	1	0	0	0	1
0	0	0	1	0	1	0	0	0
0	1	0	1	1	0	1	0	0
0	1	1	1	0	1	0	0	0
0	0	0	0	0	0	0	1	0
1	0	0	0	0	0	0	0	0
0	1	0	0	1	1	0	0	1

- rule 0: the origin $(0,0)$ is active
- rule 1: a vertex of value 1 with at least one active neighbor may become active.

Let $\mathcal{C} = (c_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \{0,1\}^{\mathbb{Z}^2}$ be a **configuration** of the square lattice $\mathbb{Z}^2$.

0	0	0	1	0	1	0	0	0
0	0	0	0	0	0	0	1	1
0	0	0	0	1	0	0	0	1
0	0	0	1	0	1	0	0	0
0	1	0	1	1	0	1	0	0
0	1	1	1	0	1	0	0	0
0	0	0	0	0	0	0	1	0
1	0	0	0	0	0	0	0	0
0	1	0	0	1	1	0	0	1

- rule 0: the origin $(0,0)$ is active
- rule 1: a vertex of value 1 with at least one active neighbor may become active.
- rule 2: a vertex of value 0 with at least two active neighbors may become active.

Let $\mathcal{C} = (c_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \{0,1\}^{\mathbb{Z}^2}$ be a **configuration** of the square lattice $\mathbb{Z}^2$.

0	0	0	1	0	1	0	0	0
0	0	0	0	0	0	0	1	1
0	0	0	0	1	0	0	0	1
0	0	0	1	0	1	0	0	0
0	1	0	1	1	0	1	0	0
0	1	1	1	0	1	0	0	0
0	0	0	0	0	0	0	1	0
1	0	0	0	0	0	0	0	0
0	1	0	0	1	1	0	0	1

- rule 0: the origin $(0,0)$ is active
- rule 1: a vertex of value 1 with at least one active neighbor may become active.
- rule 2: a vertex of value 0 with at least two active neighbors may become active.

The process terminates if no vertex can become active.

Let $\mathcal{C} = (c_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \{0,1\}^{\mathbb{Z}^2}$ be a **configuration** of the square lattice $\mathbb{Z}^2$.

0	0	0	1	0	1	0	0	0
0	0	0	0	0	0	0	1	1
0	0	0	0	1	0	0	0	1
0	0	0	1	0	1	0	0	0
0	1	0	1	1	0	1	0	0
0	1	1	1	0	1	0	0	0
0	0	0	0	0	0	0	1	0
1	0	0	0	0	0	0	0	0
0	1	0	0	1	1	0	0	1

- rule 0: the origin $(0,0)$ is active
- rule 1: a vertex of value 1 with at least one active neighbor may become active.
- rule 2: a vertex of value 0 with at least two active neighbors may become active.

The process terminates if no vertex can become active.

Let $\mathcal{C} = (c_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \{0,1\}^{\mathbb{Z}^2}$ be a **configuration** of the square lattice $\mathbb{Z}^2$.

0	0	0	1	0	1	0	0	0
0	0	0	0	0	0	0	1	1
0	0	0	0	1	0	0	0	1
0	0	0	1	0	1	0	0	0
0	1	0	1	1	0	1	0	0
0	1	1	1	0	1	0	0	0
0	0	0	0	0	0	0	1	0
1	0	0	0	0	0	0	0	0
0	1	0	0	1	1	0	0	1

- rule 0: the origin $(0,0)$ is active
- rule 1: a vertex of value 1 with at least one active neighbor may become active.
- rule 2: a vertex of value 0 with at least two active neighbors may become active.

The process terminates if no vertex can become active.

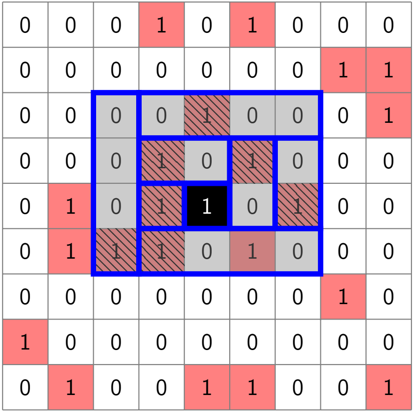
Let $\mathcal{C} = (c_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \{0,1\}^{\mathbb{Z}^2}$ be a **configuration** of the square lattice $\mathbb{Z}^2$.

0	0	0	1	0	1	0	0	0
0	0	0	0	0	0	0	1	1
0	0	0	0	1	0	0	0	1
0	0	0	1	0	1	0	0	0
0	1	0	1	1	0	1	0	0
0	1	1	1	0	1	0	0	0
0	0	0	0	0	0	0	1	0
1	0	0	0	0	0	0	0	0
0	1	0	0	1	1	0	0	1

- rule 0: the origin $(0,0)$ is active
- rule 1: a vertex of value 1 with at least one active neighbor may become active.
- rule 2: a vertex of value 0 with at least two active neighbors may become active.

The process terminates if no vertex can become active.

Let $\mathcal{C} = (c_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \{0,1\}^{\mathbb{Z}^2}$ be a **configuration** of the square lattice $\mathbb{Z}^2$.



- rule 0: the origin $(0,0)$ is active
- rule 1: a vertex of value 1 with at least one active neighbor may become active.
- rule 2: a vertex of value 0 with at least two active neighbors may become active.

The process terminates if no vertex can become active.

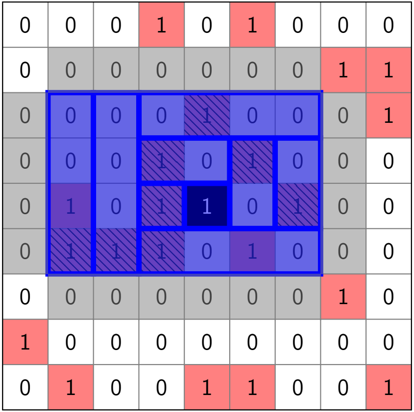
Let $\mathcal{C} = (c_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \{0,1\}^{\mathbb{Z}^2}$ be a **configuration** of the square lattice $\mathbb{Z}^2$.

0	0	0	1	0	1	0	0	0
0	0	0	0	0	0	0	1	1
0	0	0	1	0	0	0	0	1
0	0	0	1	0	1	0	0	0
0	1	0	1	1	0	1	0	0
0	1	1	1	0	1	0	0	0
0	0	0	0	0	0	0	1	0
1	0	0	0	0	0	0	0	0
0	1	0	0	1	1	0	0	1

- rule 0: the origin $(0,0)$ is active
- rule 1: a vertex of value 1 with at least one active neighbor may become active.
- rule 2: a vertex of value 0 with at least two active neighbors may become active.

The process terminates if no vertex can become active.

Let $\mathcal{C} = (c_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \{0,1\}^{\mathbb{Z}^2}$ be a **configuration** of the square lattice $\mathbb{Z}^2$.



- rule 0: the origin $(0,0)$ is active
- rule 1: a vertex of value 1 with at least one active neighbor may become active.
- rule 2: a vertex of value 0 with at least two active neighbors may become active.

The process terminates if no vertex can become active.

Fact: If the process terminates, the set of active vertices $R(\mathcal{C})$ has rectangular shape.

Let $\mathcal{C} = (c_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \{0,1\}^{\mathbb{Z}^2}$ be a **configuration** of the square lattice $\mathbb{Z}^2$.

0	0	0	1	0	1	0	0	0
0	0	0	0	0	0	0	1	1
0	0	0	0	1	0	0	0	1
0	0	0	1	0	1	0	0	0
0	1	0	1	1	0	1	0	0
0	1	1	1	0	1	0	0	0
0	0	0	0	0	0	0	1	0
1	0	0	0	0	0	0	0	0
0	1	0	0	1	1	0	0	1

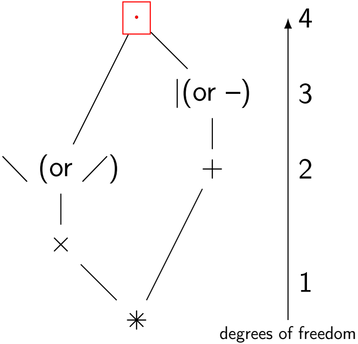
- rule 0: the origin $(0,0)$ is active
- rule 1: a vertex of value 1 with at least one active neighbor may become active.
- rule 2: a vertex of value 0 with at least two active neighbors may become active.

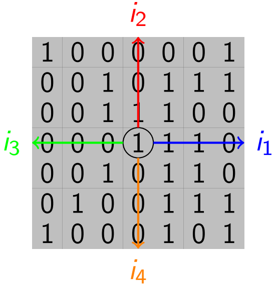
The process terminates if no vertex can become active.

Fact: If the process terminates, the set of active vertices $R(\mathcal{C})$ has rectangular shape.

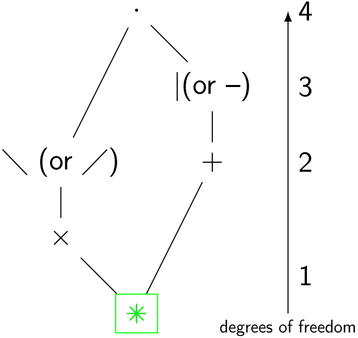
Let $(c_{i,j})$ be iid random variable $\mathbb{P}[c_{i,j} = 1] = p = 1 - q$,
$\mathbb{P}[c_{i,j} = 0] = 1 - p = q$

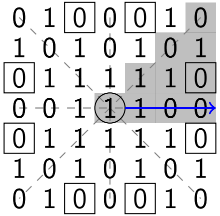
•What is the law of semi perimeter $|R(\mathcal{C})|$ of the rectangle ?



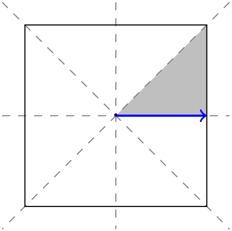


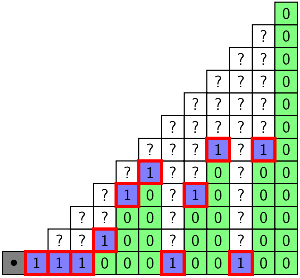
System with 9 q-linear equations





Counted by permutations





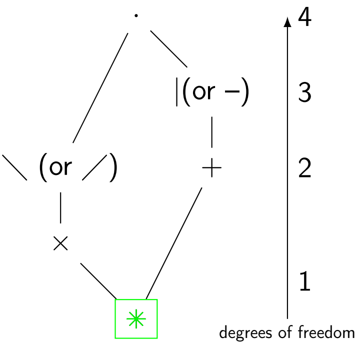
- Inversion table for some permutation.

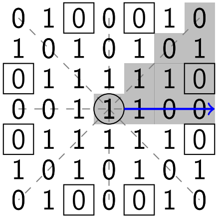
  Here : (1 6 5 9 2 11 8 4 3 7 10) with 21 inversions

For each $\mathcal{C}$, weight q^K where K is mainly the number of 0's read by an algorithm.

Proposition

$$\mathbb{P}[|R(\mathcal{C})| = 2(2n+1)] = q^{n+1}(1-q)^n \sum_{\sigma \in \mathcal{S}_n} q^{\#\textit{inversions of } \sigma}$$

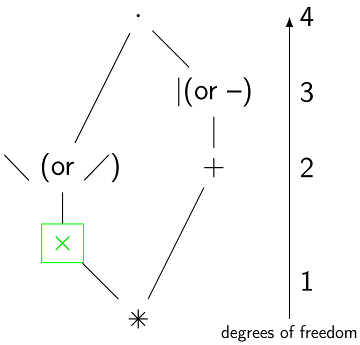


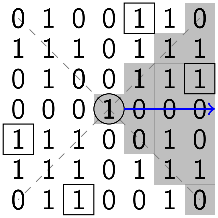


Counted by permutations

Proposition

$$\mathbb{P}[|R(\mathcal{C})| = 2(2n+1)] = q^{n+1}(1-q)^n \sum_{\sigma \in \mathcal{S}_n} q^{\#\textit{inversions of } \sigma}$$



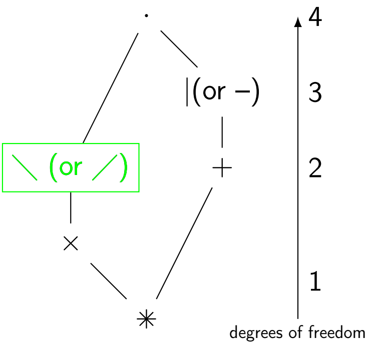


Counted by involutions

Proposition

$$\mathbb{P}[|R(\mathcal{C})| = 2(2n+1)] = q^{2n+1}(1-q)^n \sum_{\sigma \in \mathcal{I}_{2n}} q^{\frac{inv\ \sigma - n}{2}}$$

where $\mathcal{I}_{2n}$ is the set of fixed point free involutions of $\mathcal{S}_{2n}$

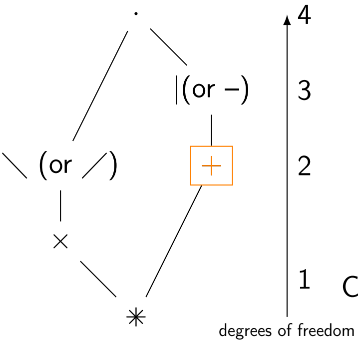


$$\begin{matrix} 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & \boxed{0} \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & \boxed{0} & 1 & 0 & 1 & 0 \end{matrix}$$

Counted by partial permutations

Theorem (D.)

$$\mathbb{P}[|R(\mathcal{C})| = 2(n+1)] = q^{2n+2}(1-q)^n \sum_{\sigma \in \mathcal{S}_n} (\#\textit{decreasing subsequences of } \sigma)\, q^{\#\textit{inversions of } \sigma}$$

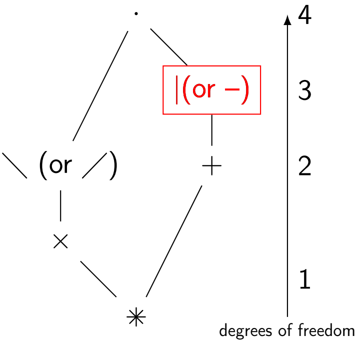


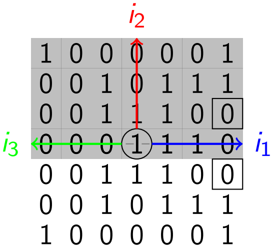
$$\begin{array}{ccccccccc}
 & & & i_2 & & & & \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & \\
\boxed{1} & 1 & 1 & 0 & 1 & 1 & \boxed{1} & \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & i_1 \\
\boxed{1} & 1 & 1 & 0 & 1 & 1 & \boxed{1} & \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & \\
0 & 0 & 1 & 0 & 1 & 0 & 0 &
\end{array}$$

Counted by the worst cases of Homing Sort [EW09, Elizalde,Winkler]

$$\begin{cases} A_{i_1,i_2} = \left(-q^{i_2+1}+1\right) A_{i_1-1,i_2} + \left((q-1)\left(q^{i_1}-1\right)\right) C_{i_1-1,i_2-1} \\ C_{i_1,i_2} = \left(q^{i_2+1}\right) A_{i_1,i_2} + \left(-\left(q^{i_1+1}-1\right)q\right) C_{i_1,i_2-1} \\ G_{i_1,i_2} = \left(q^{i_1+1}\right) C_{i_1,i_2} \end{cases}$$

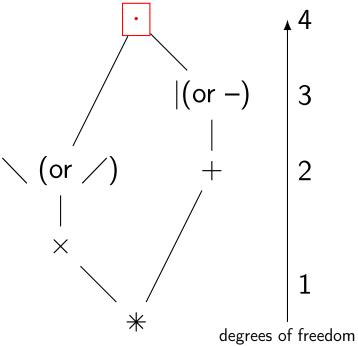
$$\mathbb{P}[|R(\mathcal{C})| = n+2] = \sum_{i_1+i_2=n} G_{i_1,i_2}$$

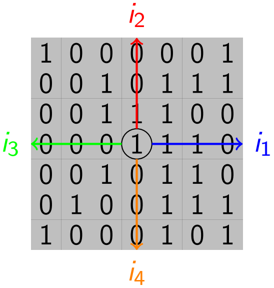




System with 4 q-linear equations

$$\begin{cases} O_{i_1,i_2,i_3} = \left(q^{i_2+1}\right) G_{i_1,i_2,i_3} \\ G_{i_1,i_2,i_3} = \left(-\left(q^{i_2+1}-1\right)q\right) G_{i_1,i_2,i_3-1} + \left(q^{i_1+i_3+1}\right) C_{i_1,i_2,i_3} \\ A_{i_1,i_2,i_3} = \left(-(q-1)^2\left(q^{i_2}-1\right)\right) G_{i_1-1,i_2-1,i_3-1} + \left(-q^{i_2+1}+1\right) A_{i_1-1,i_2,i_3} + \left((q-1)\left(q^{i_1+i_3}-1\right)\right) C_{i_1-1,i_2-} \\ C_{i_1,i_2,i_3} = \left(-\left(q^{i_1+i_3+1}-1\right)q\right) C_{i_1,i_2-1,i_3} + \left(q^{i_2+1}\right) A_{i_1,i_2,i_3} + \left((q-1)\left(q^{i_2}-1\right)q\right) G_{i_1,i_2-1,i_3-1} \end{cases}$$





System with 9 q-linear equations

Simplified sandpile model on grid graph

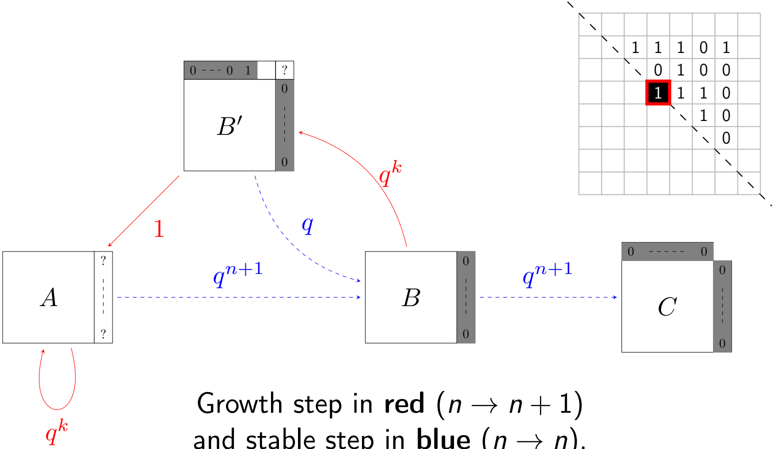
- Froböse local model: Bootstrap percolation [GH08]
- Simplification: symmetry on the sampling of $C(i, j)$
- Probabilities described by permutation statistics

From rectangular growth to partial permutations

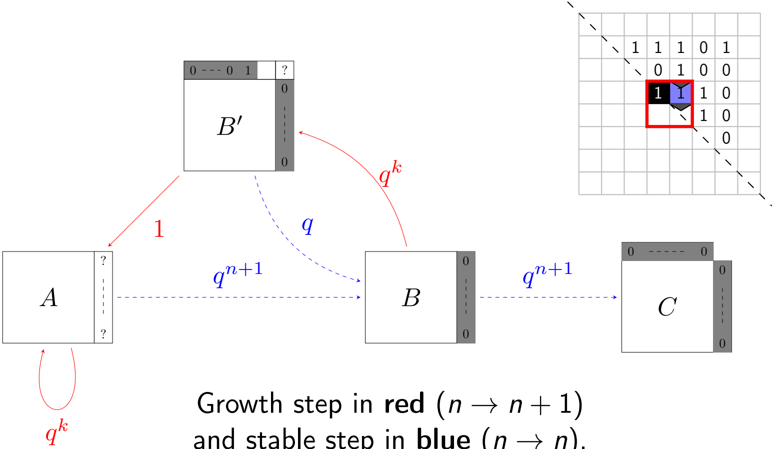
- Counter automaton for a canonical growth
- Counter automaton for partial permutations
- Automata identification

Decreasing subsequences and q-positivity

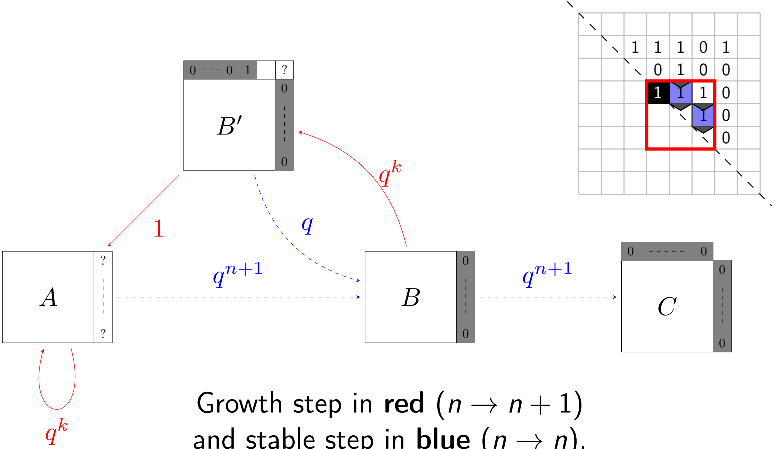
- $\sum_{\sigma \in ??} \frac{q^{\text{maj } \sigma} - q^{\text{inv } \sigma}}{1-q} \in \mathbb{N}[q]$
- Proof via Kadell weighted inversion number



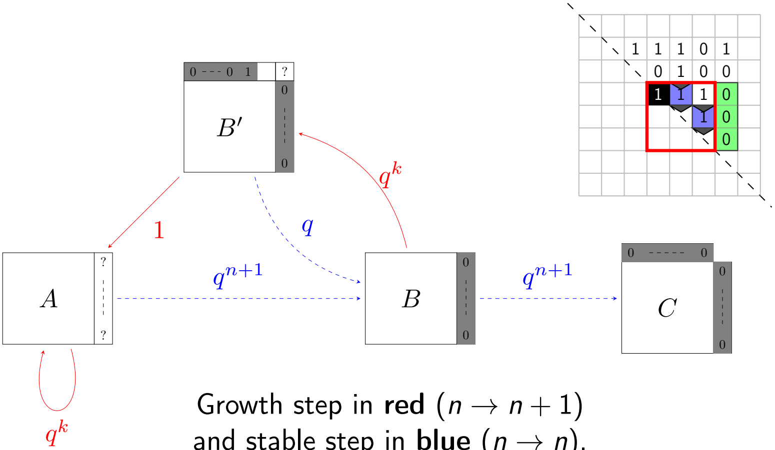
Growth step in **red** ($n \to n+1$)
and stable step in **blue** ($n \to n$).



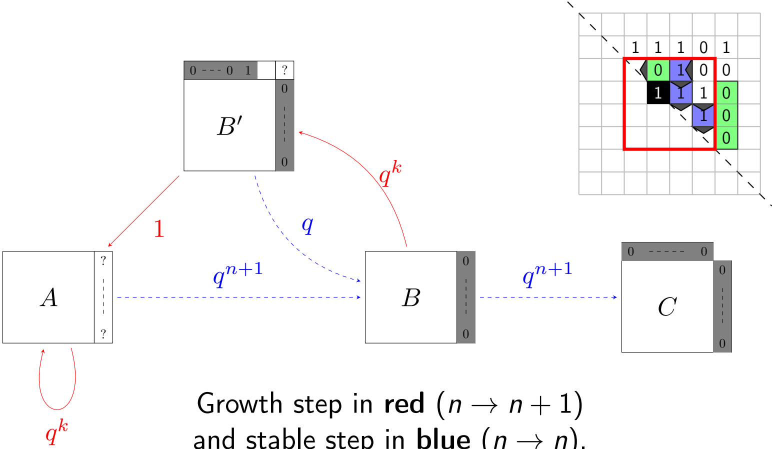
Growth step in **red** ($n \to n+1$)
and stable step in **blue** ($n \to n$).



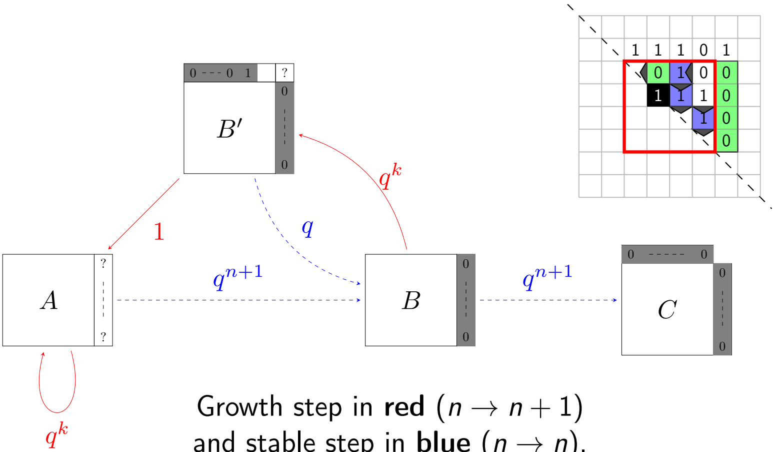
Growth step in **red** ($n \to n+1$)
and stable step in **blue** ($n \to n$).



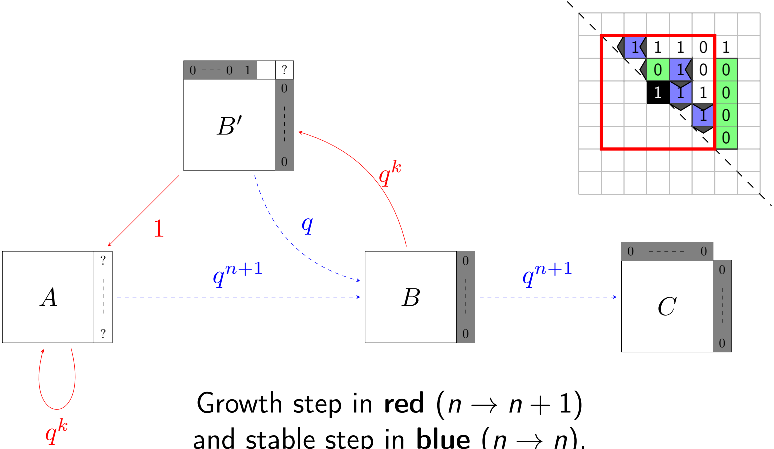
Growth step in **red** ($n \rightarrow n+1$)
and stable step in **blue** ($n \rightarrow n$).



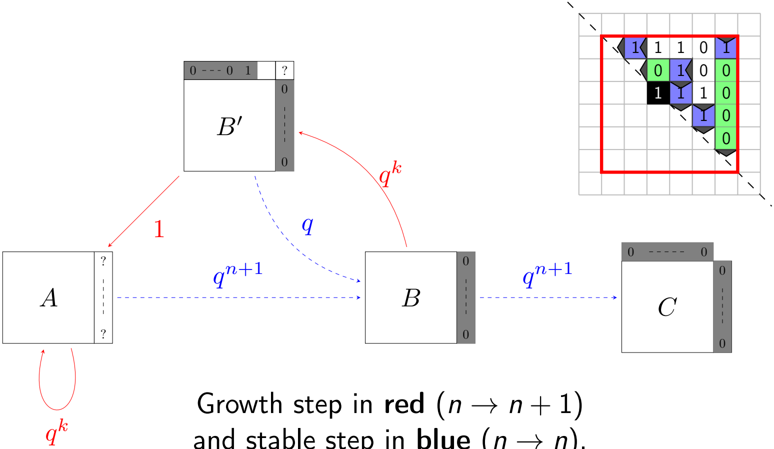
Growth step in **red** ($n \to n+1$)
and stable step in **blue** ($n \to n$).



Growth step in **red** ($n \to n+1$)
and stable step in **blue** ($n \to n$).

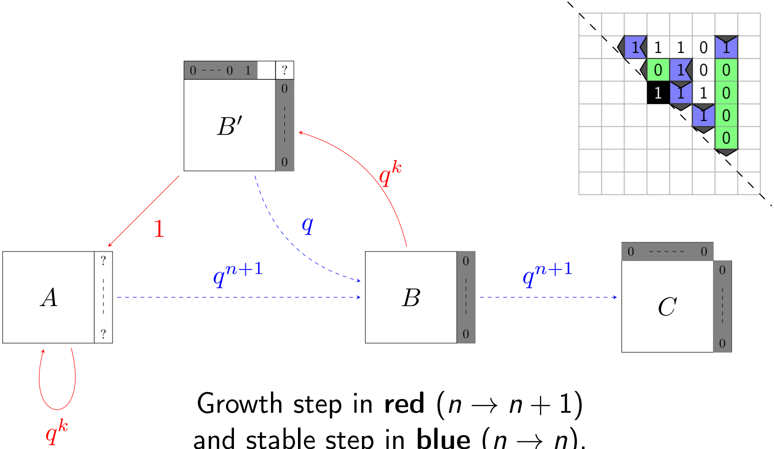


Growth step in **red** ($n \to n+1$)
and stable step in **blue** ($n \to n$).



Growth step in **red** ($n \to n+1$)
and stable step in **blue** ($n \to n$).

$$\begin{cases} A_{i_1,i_2} = (1 - q^{i_1+i_2})A_{i_1-1,i_2} + (1-q)B'_{i_1-1,i_2} \\ B'_{i_1,i_2} = \left(1 - q^{i_1+i_2}\right) B_{i_1,i_2-1} \\ B_{i_1,i_2} = (q^{i_1+i_2+1})A_{i_1,i_2} + qB'_{i_1,i_2 1} \\ C_{i_1,i_2} = (q^{i_1+i_2+1})B_{i_1,i_2} \end{cases}$$



Growth step in **red** $(n \to n+1)$
and stable step in **blue** $(n \to n)$.

$$\frac{1}{(1-q)^n q^{2n+2}} \sum_{i_1+i_2=n} C_{i_1,i_2} := C_n = 2\frac{1-q^n}{1-q} C_{n-1} - \left(\frac{1-q^{n-1}}{1-q}\right)^2 C_{n-2}$$

About partial permutations

Definition

A partial permutation is a injective partial self-maps on $\{1, \ldots, n\}$. We note R_n the set of partial permutations of size n.

$$|R_n| = 2n|R_{n-1}| - (n-1)^2|R_{n-2}| \text{ [BRR89, } \textit{Borwein, Rankin, Renner}\text{]}$$

Example

23?84?76 is a partial maps defined on $\{1, 2, 4, 5, 7, 8\}$ and undefined on $\{3, 6\}$.

Partial permutations $\leftrightarrow$ permutations with a decreasing subsequence marked.

Ex: 23?84?76 $\rightarrow$ **23**$\underline{5}$**84**$\underline{1}$**76** has 10 inversions.

$$r_n = \sum_{\sigma \in R_n} q^{\mathrm{inv}\ \sigma}$$

$\sigma = \sigma_1 \ldots \sigma_n \in R_n$

π_1 Let $k \in \{1 \ldots n+1\}$, we shift the value of σ that are greater or equal to k and we insert k at the first position.
Example: $\sigma = \underline{5}\mathbf{26}\underline{4}\mathbf{1}\underline{3}$, the rule π_1 with $k = 2$ leads to $\mathbf{2}\underline{6}\mathbf{37}\underline{5}\mathbf{1}\underline{4}$

$\sigma = \sigma_1 \dots \sigma_n \in R_n$

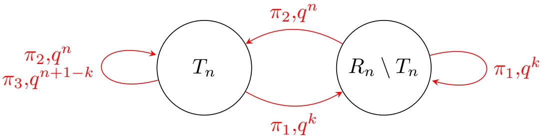
- π_1 Let $k \in \{1 \dots n+1\}$, we shift the value of σ that are greater or equal to k and we insert k at the first position.
  Example: $\sigma = \underline{5}\mathbf{26}\underline{4}\mathbf{1}\underline{3}$, the rule π_1 with $k = 2$ leads to $\mathbf{2}\underline{6}\mathbf{37}\underline{5}\mathbf{1}\underline{4}$
- π_2 We insert a marked $\underline{n+1}$ at the first position.
  Example: $\sigma = \underline{5}\mathbf{26}\underline{4}\mathbf{1}\underline{3}$, the rule π_2 leads to $\underline{65}\mathbf{26}\underline{4}\mathbf{1}\underline{3}$

$\sigma = \sigma_1 \dots \sigma_n \in R_n$

- π_1 Let $k \in \{1 \dots n+1\}$, we shift the value of σ that are greater or equal to k and we insert k at the first position.
  Example: $\sigma = \underline{5}\mathbf{26}\underline{4}\mathbf{1}\underline{3}$, the rule π_1 with $k = 2$ leads to $\mathbf{2}\underline{6}\mathbf{37}\underline{5}\mathbf{1}\underline{4}$
- π_2 We insert a marked $\underline{n+1}$ at the first position.
  Example: $\sigma = \underline{5}\mathbf{26}\underline{4}\mathbf{1}\underline{3}$, the rule π_2 leads to $\underline{65}\mathbf{26}\underline{4}\mathbf{1}\underline{3}$
- π_3 Let $k \in \{2 \dots n+1\}$, if σ_1 is marked, we insert $n+1$ at position k in σ.
  Example: $\sigma = \underline{5}\mathbf{26}\underline{4}\mathbf{1}\underline{3}$, the rule π_3 with $k = 4$ leads to $\underline{5}\mathbf{267}\underline{4}\mathbf{1}\underline{3}$

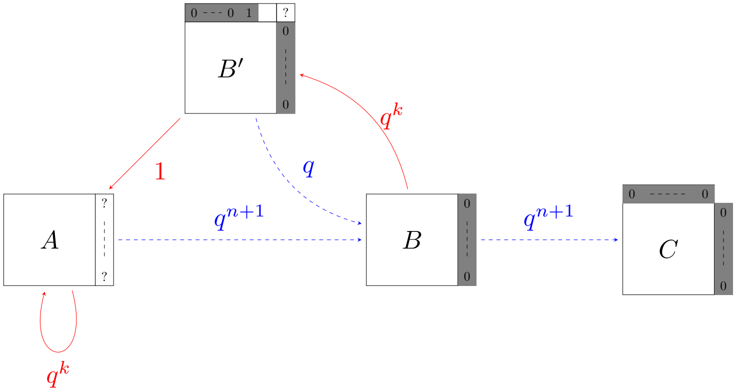
$\sigma = \sigma_1 \ldots \sigma_n \in R_n$

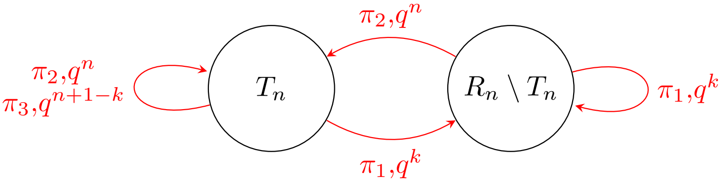
- π_1 Let $k \in \{1 \ldots n+1\}$, we shift the value of σ that are greater or equal to k and we insert k at the first position.
  Example: $\sigma = \underline{5}\mathbf{26}\underline{4}\mathbf{1}\underline{3}$, the rule π_1 with $k = 2$ leads to $\mathbf{2}\underline{6}\mathbf{37}\underline{5}\mathbf{1}\underline{4}$
- π_2 We insert a marked $\underline{n+1}$ at the first position.
  Example: $\sigma = \underline{5}\mathbf{26}\underline{4}\mathbf{1}\underline{3}$, the rule π_2 leads to $\underline{65}\mathbf{26}\underline{4}\mathbf{1}\underline{3}$
- π_3 Let $k \in \{2 \ldots n+1\}$, if σ_1 is marked, we insert $n+1$ at position k in σ.
  Example: $\sigma = \underline{5}\mathbf{26}\underline{4}\mathbf{1}\underline{3}$, the rule π_3 with $k = 4$ leads to $\underline{5}\mathbf{267}\underline{4}\mathbf{1}\underline{3}$

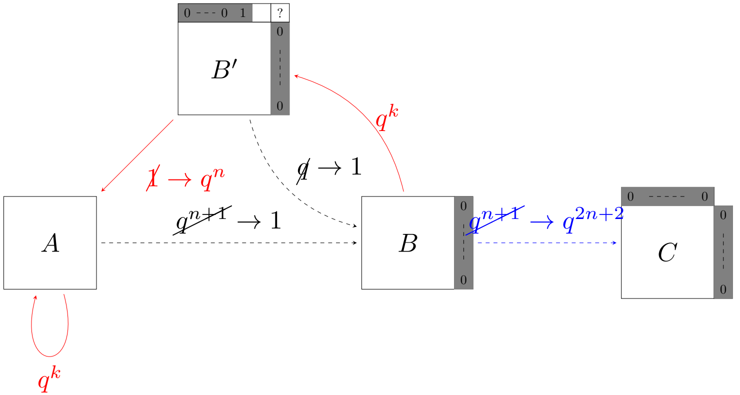


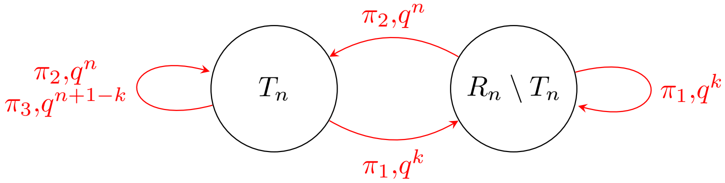
- R_n partial permutations
- T_n partial permutations undefined on 1.

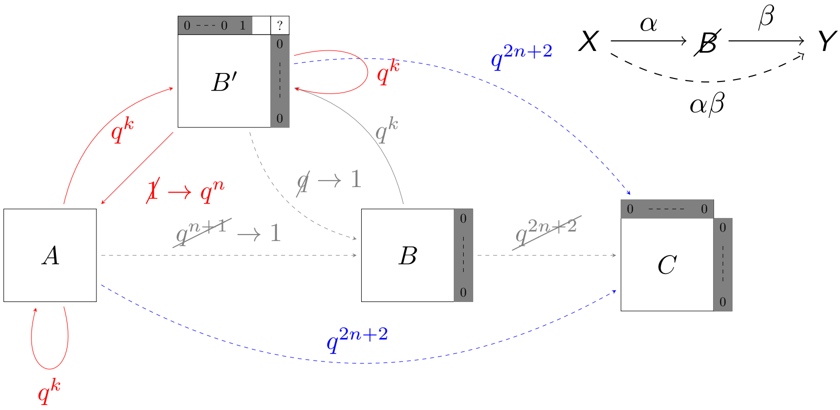
$$r_n = 2\frac{1-q^n}{1-q} r_{n-1} - \left(\frac{1-q^{n-1}}{1-q}\right)^2 r_{n-2}$$

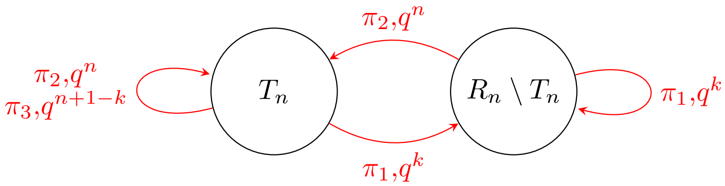

0 - - - 0 1
?
0
0
B'
q^k
1
q
A
?
?
q^{n+1}
B
0
0
q^{n+1}
0 - - - - - 0
C
0
0
q^k


π_2, q^n
π_2, q^n
π_3, q^{n+1-k}
T_n
$R_n \setminus T_n$
π_1, q^k
π_1, q^k


0 - - - 0 1 ?
0
0
B'
q^k
$\cancel{1} \to q^n$
$\cancel{q} \to 1$
A
$\cancel{q^{n+1}} \to 1$
B
0
0
$\cancel{q^{n+1}} \to q^{2n+2}$
0 - - - - - 0
0
0
C
q^k


π_2, q^n
π_2, q^n
π_3, q^{n+1-k}
T_n
$R_n \setminus T_n$
π_1, q^k
π_1, q^k


B'
A
B
C
q^k
q^k
q^k
q^k
$\cancel{1} \to q^n$
$\cancel{q} \to 1$
$\cancel{q^{n+1}} \to 1$
$\cancel{q^{2n+2}}$
q^{2n+2}
q^{2n+2}
$X \xrightarrow{\alpha} \cancel{B} \xrightarrow{\beta} Y$
$\alpha\beta$


T_n
$R_n \setminus T_n$
π_2, q^n
π_3, q^{n+1-k}
π_2, q^n
π_1, q^k
π_1, q^k

Theorem

$\mathbb{P}[|R(\mathcal{C})| = 2(n+1)] = q^{2n+2}(1-q)^n \sum_{\sigma \in R_n} q^{inv\ \sigma}$

- $\displaystyle\sum_{\sigma \in R_n} q^{\mathrm{inv}\ \sigma} = \sum_{p \prec s_n} \sum_{\substack{\sigma \in \mathcal{S}_n \\ p \prec \sigma}} q^{\mathrm{inv}\ \sigma}$ where $s_n = (n, n-1, \ldots, 1)$ and $a \prec b$ means "a is a subsequence of b".

Theorem

$\mathbb{P}[|R(\mathcal{C})| = 2(n+1)] = q^{2n+2}(1-q)^n \sum_{\sigma \in R_n} q^{inv\ \sigma}$

- $\displaystyle\sum_{\sigma \in R_n} q^{\text{inv }\sigma} = \sum_{p \prec s_n} \sum_{\substack{\sigma \in \mathcal{S}_n \\ p \prec \sigma}} q^{\text{inv }\sigma}$ where $s_n = (n, n-1, \ldots, 1)$ and $a \prec b$ means "a is a subsequence of b".
- $\sum_{\substack{\sigma \in \mathcal{S}_n \\ p \prec \sigma}} q^{\text{inv }\sigma} = q^{|p|(|p|-1)/2} \frac{[n]_q!}{[|p|]_q!}$ when $p = (\max p, \max p - 1, \ldots, \min p)$ [BW89, Bjorner, Wachs]
- $\sum_{\substack{\sigma \in \mathcal{S}_n \\ p \prec \sigma}} q^{\text{maj }\sigma} = q^{|p|(|p|-1)/2} \frac{[n]_q!}{[|p|]_q!}$ for any p.

Theorem

$\mathbb{P}[|R(\mathcal{C})| = 2(n+1)] = q^{2n+2}(1-q)^n \sum_{\sigma\in R_n} q^{inv\ \sigma}$

- $\displaystyle\sum_{\sigma\in R_n} q^{\mathrm{inv}\ \sigma} = \sum_{p\prec s_n} \sum_{\substack{\sigma\in\mathcal{S}_n \\ p\prec\sigma}} q^{\mathrm{inv}\ \sigma}$ where $s_n = (n, n-1, \ldots, 1)$ and $a \prec b$ means "a is a subsequence of b".
- $\sum_{\substack{\sigma\in\mathcal{S}_n \\ p\prec\sigma}} q^{\mathrm{inv}\ \sigma} = q^{|p|(|p|-1)/2}\frac{[n]_q!}{[|p|]_q!}$ when $p = (\max p, \max p - 1, \ldots, \min p)$ [BW89, Bjorner, Wachs]
- $\sum_{\substack{\sigma\in\mathcal{S}_n \\ p\prec\sigma}} q^{\mathrm{maj}\ \sigma} = q^{|p|(|p|-1)/2}\frac{[n]_q!}{[|p|]_q!}$ for any p.

$$\underbrace{\sum_{\substack{\sigma\in\mathcal{S}_n \\ p\prec\sigma}} q^{\mathrm{inv}\ \sigma}}_{\text{complex}} = \underbrace{\sum_{\substack{\sigma\in\mathcal{S}_n \\ p\prec\sigma}} q^{\mathrm{maj}\ \sigma}}_{\text{simple}} - \underbrace{\sum_{\substack{\sigma\in\mathcal{S}_n \\ p\prec\sigma}} \left(q^{\mathrm{maj}\ \sigma} - q^{\mathrm{inv}\ \sigma}\right)}_{\text{may be simpler}}$$

Let $\mathcal{S}_n$ the set of permutations on $\{1 \dots n\}$, s_n the decreasing permutation $(n, n-1, \dots 1)$.

Theorem (D. 2018)

For any $n \in \mathbb{N}$, $p \prec s_n$, the following sum is a polynomial with ***positive*** *integer coefficients.*

$$\sum_{\sigma \in \mathcal{S}_n^p} \frac{q^{maj\ \sigma} - q^{inv\ \sigma}}{1-q}$$

where $\mathcal{S}_n^p$ the set of permutations of $\mathcal{S}_n$ wich contain the subsequence p.

Let $\mathcal{S}_n$ the set of permutations on $\{1 \dots n\}$, s_n the decreasing permutation $(n, n-1, \dots 1)$.

Theorem (D. 2018)

For any $n \in \mathbb{N}$, $p \prec s_n$, the following sum is a polynomial with ***positive*** *integer coefficients.*

$$\sum_{\sigma\in\mathcal{S}_n^p} \frac{q^{\mathrm{maj}\ \sigma} - q^{\mathrm{inv}\ \sigma}}{1-q}$$

where $\mathcal{S}_n^p$ the set of permutations of $\mathcal{S}_n$ wich contain the subsequence p.

$$\underbrace{\sum_{\sigma\in\mathcal{S}_n^p} \frac{q^{\mathrm{maj}\ \sigma} - q^{\mathrm{inv}\ \sigma}}{1-q}}_{\text{positive ?}} = \underbrace{\sum_{\sigma\in\mathcal{S}_n^p} \frac{q^{\mathrm{maj}\ \sigma} - q^{\mathrm{stat}\ \sigma}}{1-q}}_{\text{positive ?}} + \underbrace{\sum_{\sigma\in\mathcal{S}_n^p} \frac{q^{\mathrm{stat}\ \sigma} - q^{\mathrm{inv}\ \sigma}}{1-q}}_{\text{positive ?}}$$

Let $W = (w_{i,j})_{n\times n}$ be an upper triangular matrix.

$$\mathrm{inv}_W(\sigma) = \sum_{1\le i<j\le n} w_{i,j}\chi(\sigma(i) > \sigma(j))$$

is the weighted inversion number of σ with respect to W [WK85, Kadell 1985].

$$W_{\mathrm{inv}} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1\\ 0 & 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad W_{\mathrm{maj}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 2 & 0 & 0\\ 0 & 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 0 & 4\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Interpolation with $W^{4,2} = \begin{array}{cc} \begin{matrix} & & & 4 & \end{matrix} & \\ \left(\begin{array}{ccc|cc} 0 & 1 & 1 & 0 & 0\\ 0 & 0 & 1 & 2 & 0\\ 0 & 0 & 0 & 1 & 0\\ \cline{1-3} 0 & 0 & 0 & 0 & 4\\ 0 & 0 & 0 & 0 & 0 \end{array}\right) & \begin{matrix} \\ 2 \\ \\ \\ \\ \end{matrix} \end{array} \Rightarrow \begin{cases} W^{n,1} = W_{\mathrm{inv}} \\ W^{2,1} = W_{\mathrm{maj}} \\ W^{i,1} = W^{i+1,i} \end{cases}$

$$\sum_{\sigma\in\mathcal{S}_n^p} \frac{q^{\text{maj }\sigma} - q^{\text{inv }\sigma}}{1-q} = \sum_{2\leq i<j\leq n} \sum_{\sigma\in\mathcal{S}_n^p} \frac{q^{\text{inv}_{W^{j,i}}(\sigma)} - q^{\text{inv}_{W^{j,i-1}}(\sigma)}}{1-q}$$

Lemma: For any $2 \leq i < j \leq n$ $\sum_{\sigma\in\mathcal{S}_n^p} \frac{q^{\text{inv}_{W^{j,i}}(\sigma)} - q^{\text{inv}_{W^{j,i-1}}(\sigma)}}{1-q} \in \mathbb{N}[q]$.

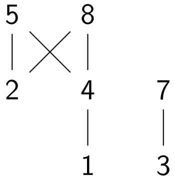
Element of the proof:

- Involution from Kadell: $*^{i,j} : \sigma \mapsto \sigma^*$ and for any σ, $\text{inv}_{W^{j,i-1}}(\sigma^*) = \text{inv}_{W^{j,i}}(\sigma)$
- If σ counts negatively, then $\sigma^* \in \mathcal{S}_n^p$

Extension of the previous lemma:

Theorem

For any $2 \leq i < j \leq n$, for any set I of inversions and set D of descent,



$$\sum_{\substack{\text{inversions of } \sigma \supset I \\ \text{desc } (\sigma^{-1}) = D}} \frac{q^{inv_{W^{j,i}}(\sigma)} - q^{inv_{W^{j,i-1}}(\sigma)}}{1-q} \in \mathbb{N}[q].$$

Remark

Related to a simplification on the sandpile model by[HJL17, Hough, Jerison, Levine]

Thank you

David Borwein, Stuart Rankin, and Lex Renner.
Enumeration of injective partial transformations.
Discrete Mathematics, 73(3):291 – 296, 1989.

Anders Björner and Michelle L Wachs.
q-hook length formulas for forests.
Journal of Combinatorial Theory, Series A, 52(2):165 – 187, 1989.

Sergi Elizalde and Peter Winkler.
Sorting by placement and shift.
In *Proceedings of the Twentieth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 68–75, Philadelphia, PA, USA, 2009.

Janko Gravner and Alexander E. Holroyd.
Local bootstrap percolation.
Electronic Journal of Probability, pages 385–399, 2008.

B. Hough, D. Jerison, and L. Levine.
Sandpiles on the square lattice.
ArXiv e-prints, March 2017.

Kevin W.J Kadell.
Weighted inversion numbers, restricted growth functions, and standard young tableaux.
40:22–44, 09 1985.