Different tree approaches to the problem of counting numerical semigroups by genus

Maria Bras-Amorós

GASCom 2018
Athens, June 19, 2018

Basic notions

- Gaps, non-gaps, genus, Frobenius number, conductor, enumeration
- Generators

Classical problems

- Frobenius' coin exchange problem
- Hurwitz question
- Wilf's conjecture

Counting

- Conjecture
- Dyck paths and Catalan bounds
- Semigroup tree and Fibonacci bounds
- Ordinarization transform and ordinarization tree
- Quasi-ordinarization transform and quasi-ordinarization forest

Basic notions

- Gaps, non-gaps, genus, Frobenius number, conductor, enumeration
- Generators

Classical problems

- Frobenius' coin exchange problem
- Hurwitz question
- Wilf's conjecture

Counting

- Conjecture
- Dyck paths and Catalan bounds
- Semigroup tree and Fibonacci bounds
- Ordinarization transform and ordinarization tree
- Quasi-ordinarization transform and quasi-ordinarization forest

Definition

A numerical semigroup is a subset Λ of $\mathbb{N}_0$ satisfying

- $0 \in \Lambda$
- $\Lambda + \Lambda \subseteq \Lambda$
- $\#(\mathbb{N}_0 \setminus \Lambda)$ is finite (genus$:=g:=\#(\mathbb{N}_0 \setminus \Lambda)$)

Definition

A numerical semigroup is a subset Λ of $\mathbb{N}_0$ satisfying

- $0 \in \Lambda$
- $\Lambda + \Lambda \subseteq \Lambda$
- $\#(\mathbb{N}_0 \setminus \Lambda)$ is finite (genus$:= g := \#(\mathbb{N}_0 \setminus \Lambda)$)

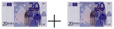
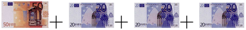
gaps: $\mathbb{N}_0 \setminus \Lambda$
non-gaps: Λ

Cash point

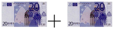
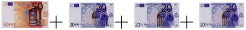
The amounts of money one can obtain from a cash point (divided by 10)

Illustration: Agnès Capella Sala

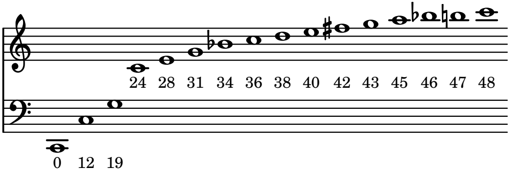
Cash point

amount		amount/10
0		0
10	*impossible*	
20		2
30	*impossible*	
40		4
50		5
60		6
70		7
80		8
90		9
100		10
110		11
⋮	⋮	⋮

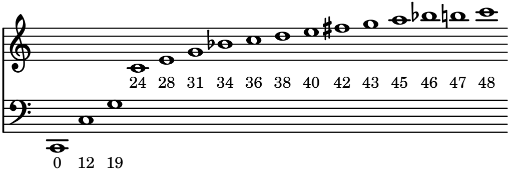
Cash point

amount		amount/10
0		0
		gap
20		2
		gap
40		4
50		5
60		6
70		7
80		8
90		9
100		10
110		11
⋮	⋮	⋮

The Well-tempered semigroup

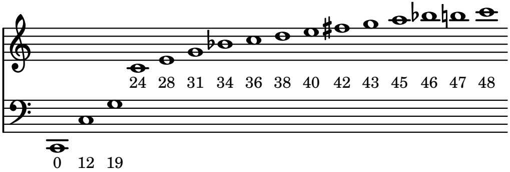


The Well-tempered semigroup



$$H = \{0, 12, 19, 24, 28, 31, 34, 36, 38, 40, 42, 43, 45, 46, 47, 48, \ldots\}$$

The Well-tempered semigroup



$$H = \{0, 12, 19, 24, 28, 31, 34, 36, 38, 40, 42, 43, 45, 46, 47, 48, \ldots\}$$

Definition

A numerical semigroup is a subset Λ of $\mathbb{N}_0$ satisfying

- $0 \in \Lambda$
- $\Lambda + \Lambda \subseteq \Lambda$
- $\#(\mathbb{N}_0 \setminus \Lambda)$ is finite (genus$:= g := \#(\mathbb{N}_0 \setminus \Lambda)$)

gaps: $\mathbb{N}_0 \setminus \Lambda$
non-gaps: Λ

The third condition implies that there exist

Definition

A numerical semigroup is a subset Λ of $\mathbb{N}_0$ satisfying

- $0 \in \Lambda$
- $\Lambda + \Lambda \subseteq \Lambda$
- $\#(\mathbb{N}_0 \setminus \Lambda)$ is finite (genus$:=g:= \#(\mathbb{N}_0 \setminus \Lambda)$)

gaps: $\mathbb{N}_0 \setminus \Lambda$
non-gaps: Λ

The third condition implies that there exist

Frobenius number $:=$ the largest gap F

Definition

A numerical semigroup is a subset Λ of $\mathbb{N}_0$ satisfying

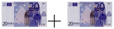
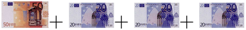
- $0 \in \Lambda$
- $\Lambda + \Lambda \subseteq \Lambda$
- $\#(\mathbb{N}_0 \setminus \Lambda)$ is finite (genus$:=$$g$$:= \#(\mathbb{N}_0 \setminus \Lambda)$)

gaps: $\mathbb{N}_0 \setminus \Lambda$
non-gaps: Λ

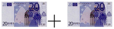
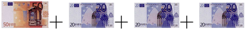
The third condition implies that there exist

Frobenius number $:=$ the largest gap F
conductor $:=$ the unique integer c with $c-1 \notin \Lambda$, $c+\mathbb{N}_0 \subseteq \Lambda$ ($c = F+1$)

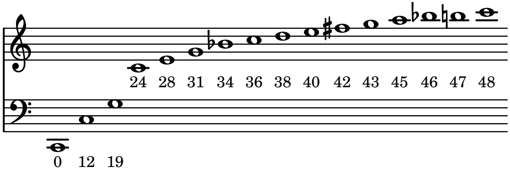
Cash point

amount		amount/10
0		0
20		2
		(3)
40		4
50		5
60		6
70		7
80		8
90		9
100		10
110		11
⋮	⋮	⋮

Cash point

amount		amount/10
0		0
20		2
40		4
50		5
60		6
70		7
80		8
90		9
100		10
110		11
⋮	⋮	⋮

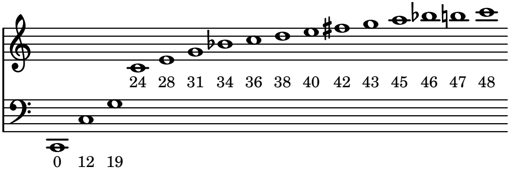
The Well-tempered semigroup



$$H = \{0, 12, 19, 24, 28, 31, 34, 36, 38, 40, 42, 43, 45, 46, 47, 48, \dots\}$$

- $g = 33$

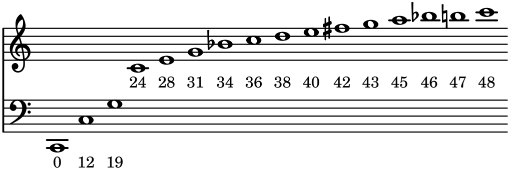
The Well-tempered semigroup



$$H = \{0, 12, 19, 24, 28, 31, 34, 36, 38, 40, 42, 43, 45, 46, 47, 48, \dots\}$$

- $g = 33$
- $c = 45$

The Well-tempered semigroup



$$H = \{0, 12, 19, 24, 28, 31, 34, 36, 38, 40, 42, 43, 45, 46, 47, 48, \ldots\}$$

- $g = 33$
- $c = 45$
- $F = 44$

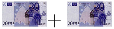
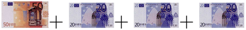
Enumeration of a numerical semigroup

The inclusion $\Lambda \subseteq \mathbb{N}_0$ implies that there exists

enumeration $:=$ the unique bijective increasing map $\lambda : \mathbb{N}_0 \to \Lambda$

$$\Lambda = \{\lambda_0 = 0 < \lambda_1 < \lambda_2 \dots\}$$

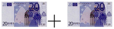
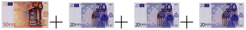
Cash point

amount		amount/10	
0		0	λ_0
20		2	λ_1
40		4	λ_2
50		5	λ_3
60		6	λ_4
70		7	λ_5
80		8	λ_6
90		9	λ_7
100		10	λ_8
110		11	λ_9
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Generators

The generators of a numerical semigroup are those non-gaps which can not be obtained as a sum of two smaller non-gaps.

Cash point

amount		amount/10
0		0
20		2
40		4
50		5
60		6
70		7
80		8
90		9
100		10
110		11
⋮	⋮	⋮

Generators

The generators of a numerical semigroup are those non-gaps which can not be obtained as a sum of two smaller non-gaps.

If $a_1, \ldots, a_l$ are the generators of a semigroup Λ then

$$\Lambda = \{n_1 a_1 + \cdots + n_l a_l : n_1, \ldots, n_l \in \mathbb{N}_0\}$$

Generators

The generators of a numerical semigroup are those non-gaps which can not be obtained as a sum of two smaller non-gaps.

If $a_1, \dots, a_l$ are the generators of a semigroup Λ then

$$\Lambda = \{n_1 a_1 + \cdots + n_l a_l : n_1, \dots, n_l \in \mathbb{N}_0\}$$

So, $a_1, \dots, a_l$ are necessarily coprime.

Generators

The generators of a numerical semigroup are those non-gaps which can not be obtained as a sum of two smaller non-gaps.

If $a_1, \dots, a_l$ are the generators of a semigroup Λ then

$$\Lambda = \{n_1 a_1 + \cdots + n_l a_l : n_1, \dots, n_l \in \mathbb{N}_0\}$$

So, $a_1, \dots, a_l$ are necessarily coprime.

If $a_1, \dots, a_l$ are coprime we define the semigroup generated by $a_1, \dots, a_l$ as

$$\langle a_1, \dots, a_n \rangle := \{n_1 a_1 + \cdots + n_l a_l : n_1, \dots, n_l \in \mathbb{N}_0\}.$$

Basic notions

- Gaps, non-gaps, genus, Frobenius number, conductor, enumeration
- Generators

Classical problems

- Frobenius' coin exchange problem
- Hurwitz question
- Wilf's conjecture

Counting

- Conjecture
- Dyck paths and Catalan bounds
- Semigroup tree and Fibonacci bounds
- Ordinarization transform and ordinarization tree
- Quasi-ordinarization transform and quasi-ordinarization forest

Frobenius' problem

What is the largest monetary amount that can not be obtained using only coins of specified denominations $a_1, \ldots, a_n$.

Frobenius' problem

What is the largest monetary amount that can not be obtained using only coins of specified denominations $a_1, \ldots, a_n$.

If $a_1, \ldots, a_n$ are coprime then the set of amounts that can be obtained is the semigroup $\langle a_1, \ldots, a_n \rangle$ and the question is determining the Frobenius number.

Frobenius' problem

What is the largest monetary amount that can not be obtained using only coins of specified denominations $a_1, \ldots, a_n$.

If $a_1, \ldots, a_n$ are coprime then the set of amounts that can be obtained is the semigroup $\langle a_1, \ldots, a_n \rangle$ and the question is determining the Frobenius number.

$n = 2$: Sylvester's formula $a_1 a_2 - a_1 - a_2$.

Frobenius' problem

What is the largest monetary amount that can not be obtained using only coins of specified denominations $a_1, \ldots, a_n$.

If $a_1, \ldots, a_n$ are coprime then the set of amounts that can be obtained is the semigroup $\langle a_1, \ldots, a_n \rangle$ and the question is determining the Frobenius number.

$n = 2$: Sylvester's formula $a_1 a_2 - a_1 - a_2$.

$n > 2$?

Theorem (Curtis)

There is no finite set of polynomials $\{f_1, \ldots, f_n\}$ such that for each choice of $a_1, a_2, a_3 \in \mathbb{N}$, there is some i such that the Frobenius number of a_1, a_2, a_3 is $f_i(a_1, a_2, a_3)$.

Some refences on Frobenius' coin exchange problem:

J. L. Ramírez Alfonsín. The Diophantine Frobenius problem, volume 30 of Oxford Lecture Series in Mathematics and its Applications. Oxford University Press, Oxford, 2005.

Frank Curtis. On formulas for the Frobenius number of a numerical semi- group. Math. Scand., 67(2):190–192, 1990.

Hurwitz problems

- Determining whether there exist non-Weierstrass numerical semigroups, (Buchweitz gave a positive answer)
- Characterizing Weierstrass semigroups

Some references:

Fernando Torres. On certain N -sheeted coverings of curves and numerical semigroups which cannot be realized as Weierstrass semigroups. Comm. Algebra, 23(11):4211–4228, 1995.

Seon Jeong Kim. Semigroups which are not Weierstrass semigroups. Bull. Korean Math. Soc., 33(2):187–191, 1996.

Jiryo Komeda. Non-Weierstrass numerical semigroups. Semigroup Forum, 57(2):157–185, 1998.

N. Kaplan and L. Ye. The proportion of Weierstrass semigroups, J. Algebra 373:377–391, 2013.

Wilf's conjecture

The number e of generators of a numerical semigroup of genus g and conductor c satisfies

$$e \geqslant \frac{c}{c-g}.$$

Wilf's conjecture

The number e of generators of a numerical semigroup of genus g and conductor c satisfies

$$e \geqslant \frac{c}{c-g}.$$

Example: If $c = 2g$ (symmetric semigroups) then $\frac{c}{c-g} = \frac{2g}{g} = 2$.

Some references:

H. Wilf. A circle-of-lights algorithm for the money-changing problem, American Mathematical Monthly 85 (1978) 562–565.

D. E. Dobbs, G. L. Matthews. On a question of Wilf concerning numerical semigroups. International Journal of Commutative Rings, 3(2), 2003.

A. Zhai. An asymptotic result concerning a question of Wilf Alex Zhai, arXiv:1111.2779.

A. Sammartano. Numerical semigroups with large embedding dimension satisfy Wilf's conjecture, Semigroup Forum 85 (2012) 439–447.

N. Kaplan. Counting numerical semigroups by genus and some cases of a question of Wilf, J. Pure Appl. Algebra 216 (2012) 1016–1032.

A. Moscariello, A. Sammartano. On a conjecture by Wilf about the Frobenius number, Math. Z. 280 (2015) 47–53.

S. Eliahou. Wilf's conjecture and Macaulay's theorem. arXiv:1703.01761

M. Delgado, On a question of Eliahou and a conjecture of Wilf. arXiv:1608.01353

Wilf conjecture

For brute approach:

M. Bras-Amorós. Fibonacci-like behavior of the number of numerical semigroups of a given genus. Semigroup Forum, 76(2):379–384, 2008.

J. Fromentin, F. Hivert. Exploring the tree of numerical semigroups. Mathematics of Computation 85 (2016), no. 301, 2553–2568.

Basic notions

- Gaps, non-gaps, genus, Frobenius number, conductor, enumeration
- Generators

Classical problems

- Frobenius' coin exchange problem
- Hurwitz question
- Wilf's conjecture

Counting

- Conjecture
- Dyck paths and Catalan bounds
- Semigroup tree and Fibonacci bounds
- Ordinarization transform and ordinarization tree
- Quasi-ordinarization transform and quasi-ordinarization forest

Counting semigroups by genus

Let n_g denote the number of numerical semigroups of genus g.

Counting semigroups by genus

Let n_g denote the number of numerical semigroups of genus g.

- $n_0 = 1$, since the unique numerical semigroup of genus 0 is $\mathbb{N}_0$

Counting semigroups by genus

Let n_g denote the number of numerical semigroups of genus g.

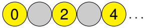
- $n_0 = 1$, since the unique numerical semigroup of genus 0 is $\mathbb{N}_0$
- $n_1 = 1$, since the unique numerical semigroup of genus 1 is

Counting semigroups by genus

Let n_g denote the number of numerical semigroups of genus g.

- $n_0 = 1$, since the unique numerical semigroup of genus 0 is $\mathbb{N}_0$
- $n_1 = 1$, since the unique numerical semigroup of genus 1 is

- $n_2 = 2$. Indeed the unique numerical semigroups of genus 2 are

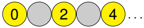


Counting semigroups by genus

Let n_g denote the number of numerical semigroups of genus g.

- $n_0 = 1$, since the unique numerical semigroup of genus 0 is $\mathbb{N}_0$
- $n_1 = 1$, since the unique numerical semigroup of genus 1 is

- $n_2 = 2$. Indeed the unique numerical semigroups of genus 2 are



- $n_3 = 4$
- $n_4 = 7$
- $n_5 = 12$
- $n_6 = 23$
- $n_7 = 39$
- $n_8 = 67$

$\vdots$

Counting semigroups by genus

Conjecture

[Bras-Amorós, 2008]

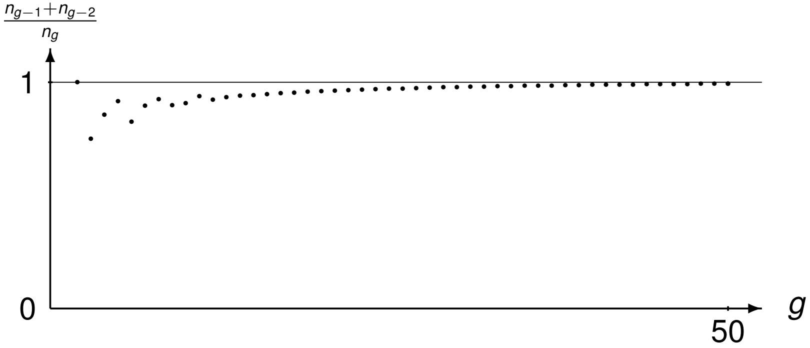
1. $n_g \geqslant n_{g-1} + n_{g-2}$
2. - $\lim_{g\to\infty} \frac{n_{g-1}+n_{g-2}}{n_g} = 1$
   - $\lim_{g\to\infty} \frac{n_g}{n_{g-1}} = \phi$

Counting semigroups by genus

g	n_g	$n_{g-1}+n_{g-2}$	$\frac{n_{g-1}+n_{g-2}}{n_g}$	$\frac{n_g}{n_{g-1}}$
0	1			
1	1			1
2	2	2	1	2
3	4	3	0.75	2
4	7	6	0.857143	1.75
5	12	11	0.916667	1.71429
6	23	19	0.826087	1.91667
7	39	35	0.897436	1.69565
8	67	62	0.925373	1.71795
9	118	106	0.898305	1.76119
10	204	185	0.906863	1.72881
11	343	322	0.938776	1.68137
12	592	547	0.923986	1.72595
13	1001	935	0.934066	1.69088
14	1693	1593	0.940933	1.69131
15	2857	2694	0.942947	1.68754
16	4806	4550	0.946733	1.68218
17	8045	7663	0.952517	1.67395
18	13467	12851	0.954259	1.67396
19	22464	21512	0.957621	1.66808
20	37396	35931	0.960825	1.66471
21	62194	59860	0.962472	1.66312
22	103246	99590	0.964589	1.66006
23	170963	165440	0.967695	1.65588
24	282828	274209	0.969526	1.65432
25	467224	453791	0.971249	1.65197
26	770832	750052	0.973042	1.64981
27	1270267	1238056	0.974642	1.64792
28	2091030	2041099	0.976121	1.64613
29	3437839	3361297	0.977735	1.64409
30	5646773	5528869	0.979120	1.64254
31	9266788	9084612	0.980341	1.64108
32	15195070	14913561	0.981474	1.63973
33	24896206	24461858	0.982554	1.63844
34	40761087	40091276	0.983567	1.63724
35	66687201	65657293	0.984556	1.63605
36	109032500	107448288	0.985470	1.63498
37	178158289	175719701	0.986312	1.63399
38	290939807	287190789	0.987114	1.63304
39	474851445	469098096	0.987884	1.63213
40	774614284	765791252	0.988610	1.63128

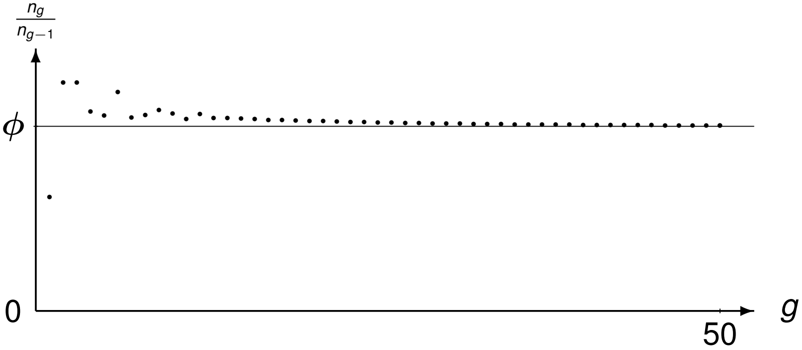
Counting semigroups by genus

Behavior of $\frac{n_{g-1}+n_{g-2}}{n_g}$



Counting semigroups by genus

Behavior of $\frac{n_g}{n_{g-1}}$



Counting semigroups by genus

What is known

- Upper and lower bounds for n_g
  Dyck paths and Catalan bounds (w. de Mier), semigroup tree and Fibonacci bounds, Elizalde's improvements, and others

Counting semigroups by genus

What is known

- Upper and lower bounds for n_g
  Dyck paths and Catalan bounds (w. de Mier), semigroup tree and Fibonacci bounds, Elizalde's improvements, and others
- $\lim_{g\to\infty} \frac{n_g}{n_{g-1}} = \phi$
  Alex Zhai (2013) with important contributions of Nathan Kaplan, Yufei Zhao, and others

Counting semigroups by genus

What is known

- Upper and lower bounds for n_g
  Dyck paths and Catalan bounds (w. de Mier), semigroup tree and Fibonacci bounds, Elizalde's improvements, and others
- $\lim_{g\to\infty} \frac{n_g}{n_{g-1}} = \phi$
  Alex Zhai (2013) with important contributions of Nathan Kaplan, Yufei Zhao, and others

Weaker unsolved conjecture

- n_g is increasing

Basic notions

Gaps, non-gaps, genus, Frobenius number, conductor, enumeration

Generators

Classical problems

Frobenius' coin exchange problem

Hurwitz question

Wilf's conjecture

Counting

Conjecture

Dyck paths and Catalan bounds

Semigroup tree and Fibonacci bounds

Ordinarization transform and ordinarization tree

Quasi-ordinarization transform and quasi-ordinarization forest

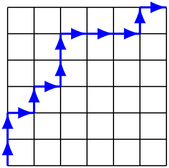
Dyck paths

A Dyck path of order n is a staircase walk from $(0, 0)$ to (n, n) that lies over the diagonal $x = y$.

Dyck paths

A Dyck path of order n is a staircase walk from $(0, 0)$ to (n, n) that lies over the diagonal $x = y$.

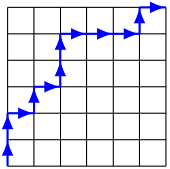
Example



Dyck paths

A Dyck path of order n is a staircase walk from $(0, 0)$ to (n, n) that lies over the diagonal $x = y$.

Example



The number of Dyck paths of order n is given by the Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Dyck paths

Definition

The square diagram of a numerical semigroup is the path

$$e(i) = \begin{cases} \rightarrow & \text{if } i \in \Lambda, \\ \uparrow & \text{if } i \notin \Lambda, \end{cases} \qquad \text{for } 1 \leqslant i \leqslant 2g.$$

Dyck paths

Definition

The square diagram of a numerical semigroup is the path

$$e(i) = \begin{cases} \rightarrow & \text{if } i \in \Lambda, \\ \uparrow & \text{if } i \notin \Lambda, \end{cases} \qquad \text{for } 1 \leqslant i \leqslant 2g.$$

It always goes from $(0,0)$ to (g,g).

Dyck paths

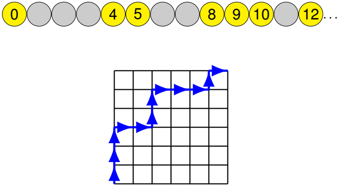
Definition

The square diagram of a numerical semigroup is the path

$$e(i) = \begin{cases} \rightarrow & \text{if } i \in \Lambda, \\ \uparrow & \text{if } i \notin \Lambda, \end{cases} \qquad \text{for } 1 \leqslant i \leqslant 2g.$$

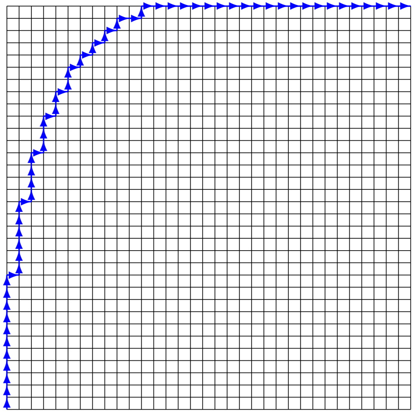
It always goes from $(0, 0)$ to (g, g).

Example



Dyck paths

Example



Dyck paths

Lemma

[Bras-Amorós, de Mier, 2007]
The square diagram of a numerical semigroup is a Dyck path.

Dyck paths

Lemma

[Bras-Amorós, de Mier, 2007]
The square diagram of a numerical semigroup is a Dyck path.

Corollary

$n_g \leqslant C_g = \frac{1}{g+1}\binom{2g}{g}$.

Basic notions

- Gaps, non-gaps, genus, Frobenius number, conductor, enumeration
- Generators

Classical problems

- Frobenius' coin exchange problem
- Hurwitz question
- Wilf's conjecture

Counting

- Conjecture
- Dyck paths and Catalan bounds
- Semigroup tree and Fibonacci bounds
- Ordinarization transform and ordinarization tree
- Quasi-ordinarization transform and quasi-ordinarization forest

Tree $\mathcal{T}$ of numerical semigroups

From genus g to genus $g-1$

A semigroup of genus g together with its Frobenius number is another semigroup of genus $g-1$.

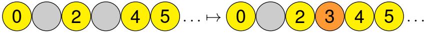
$$\Lambda \mapsto \Lambda \cup \{F(\Lambda)\}$$

Tree $\mathcal{T}$ of numerical semigroups

From genus g to genus $g-1$

A semigroup of genus g together with its Frobenius number is another semigroup of genus $g-1$.

$$\Lambda \mapsto \Lambda \cup \{F(\Lambda)\}$$

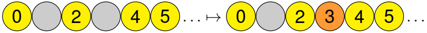


Tree $\mathcal{T}$ of numerical semigroups

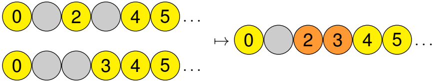
From genus g to genus $g-1$

A semigroup of genus g together with its Frobenius number is another semigroup of genus $g-1$.

$$\Lambda \mapsto \Lambda \cup \{F(\Lambda)\}$$



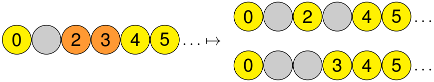
A set of semigroups may give the same semigroup when adjoining their Frobenius numbers.



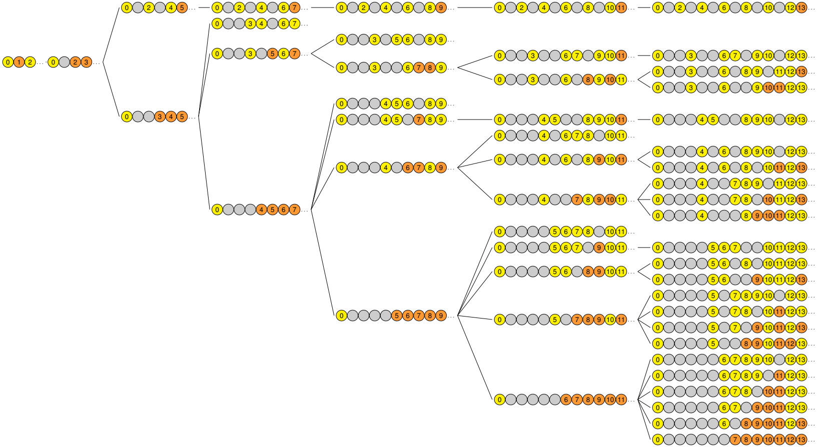
Tree $\mathcal{T}$ of numerical semigroups

From genus $g-1$ to genus g

All semigroups giving Λ when adjoining to them their Frobenius number can be obtained from Λ by taking out one by one all generators of Λ larger than its Frobenius number.

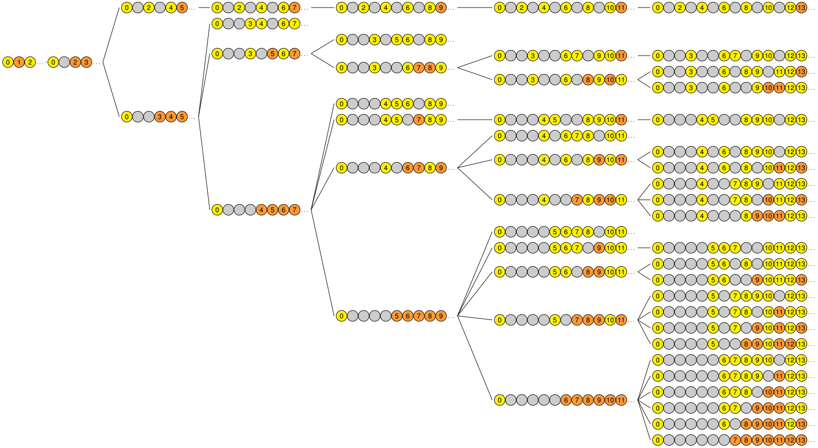


Tree $\mathcal{T}$ of numerical semigroups



The parent of a semigroup Λ is Λ together with its Frobenius number.

Tree $\mathcal{T}$ of numerical semigroups

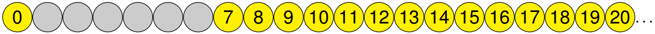


The parent of a semigroup Λ is Λ together with its Frobenius number.

The descendants of a semigroup are obtained taking away one by one all generators larger than its Frobenius number.

Tree $\mathcal{T}$ of numerical semigroups

A numerical semigroup is ordinary if all its gaps are consecutive.



Tree $\mathcal{T}$ of numerical semigroups

Descendants of ordinary semigroups

Lemma

The ordinary semigroup of genus g *has* $g+1$ *descendants which in turn have* $0, 1, 2, \ldots, g-2,\ g,\ g+2$ *descendants.*

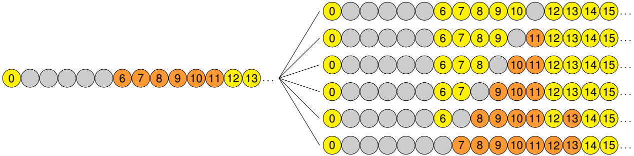
Tree $\mathcal{T}$ of numerical semigroups

Descendants of ordinary semigroups

Lemma

The ordinary semigroup of genus g has $g+1$ descendants which in turn have $0, 1, 2, \ldots, g-2,\ g,\ g+2$ *descendants.*

Example



Tree $\mathcal{T}$ of numerical semigroups

Descendants of non-ordinary semigroups

Lemma

If the generators of Λ (non-ordinary) that are larger than its Frobenius number are $\{\lambda_{i_1} < \lambda_{i_2} < \cdots < \lambda_{i_k}\}$, then the generators of $\Lambda \setminus \{\lambda_{i_j}\}$ that are larger than its Frobenius number are

$$\{\lambda_{i_{j+1}} < \cdots < \lambda_{i_k}\},$$

or

$$\{\lambda_{i_{j+1}} < \cdots < \lambda_{i_k}\} \cup \{\lambda_1 + \lambda_{i_j}\}$$

Tree $\mathcal{T}$ of numerical semigroups

Descendants of non-ordinary semigroups

Lemma

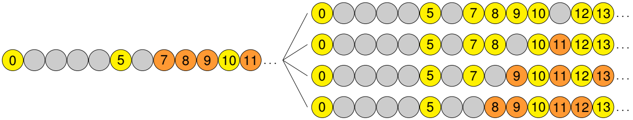
If the generators of Λ (non-ordinary) that are larger than its Frobenius number are $\{\lambda_{i_1} < \lambda_{i_2} < \cdots < \lambda_{i_k}\}$, then the generators of $\Lambda \setminus \{\lambda_{i_j}\}$ that are larger than its Frobenius number are

$$\{\lambda_{i_{j+1}} < \cdots < \lambda_{i_k}\},$$

or

$$\{\lambda_{i_{j+1}} < \cdots < \lambda_{i_k}\} \cup \{\lambda_1 + \lambda_{i_j}\}$$

Example



Tree $\mathcal{T}$ of numerical semigroups

Corollary

If a non-ordinary node in the semigroup tree has k descendants, then its descendants have

- *at least* $0, \dots, k-1$ *descendants, respectively,*
- *at most* $1, \dots, k$ *descendants, respectively.*

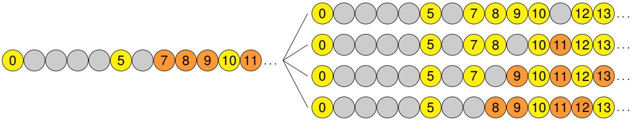
Tree $\mathcal{T}$ of numerical semigroups

Corollary

If a non-ordinary node in the semigroup tree has k descendants, then its descendants have

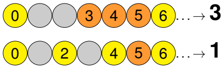
- *at least* $0, \ldots, k-1$ *descendants, respectively,*
- *at most* $1, \ldots, k$ *descendants, respectively.*

Example



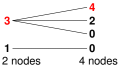
Subtree

Number of descendants of semigroups of genus 2



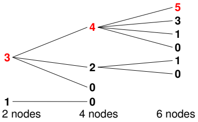
Subtree

Lower bound for the number of descendants of semigroups of genus 3



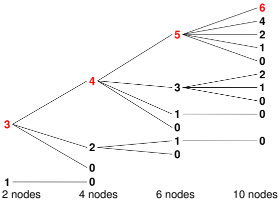
Subtree

Lower bound for the number of descendants of semigroups of genus 4



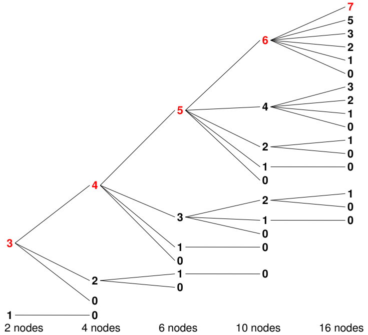
Subtree

Lower bound for the number of descendants of semigroups of genus 5



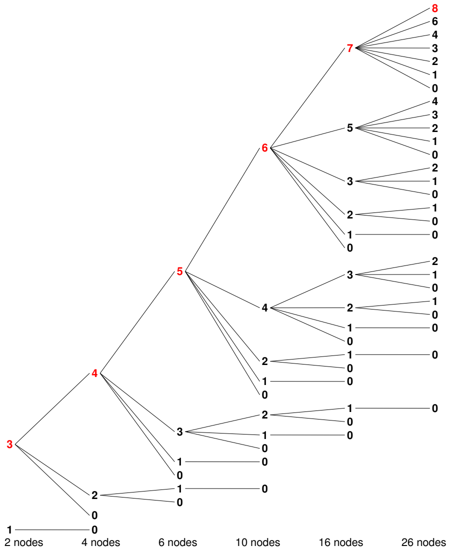
Subtree

Lower bound for the number of descendants of semigroups of genus 6



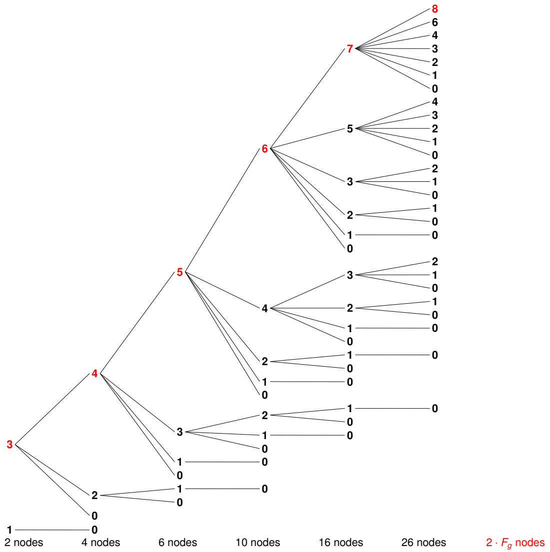
Subtree

Lower bound for the number of descendants of semigroups of genus 7



Subtree

Lower bound for the number of descendants of semigroups of genus g



Subtree

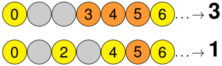
Lemma

For $g \geqslant 3$,

$$2F_g \leqslant n_g$$

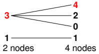
Supertree

Number of descendants of semigroups of genus 2



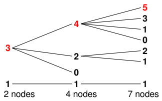
Supertree

Upper bound for the number of descendants of semigroups of genus 3



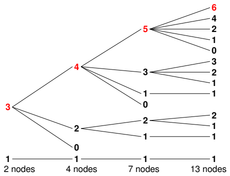
Supertree

Upper bound for the number of descendants of semigroups of genus 4



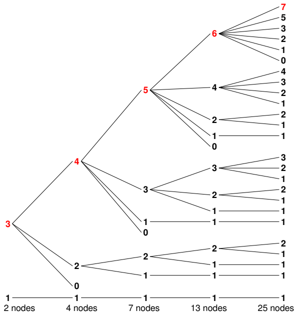
Supertree

Upper bound for the number of descendants of semigroups of genus 5



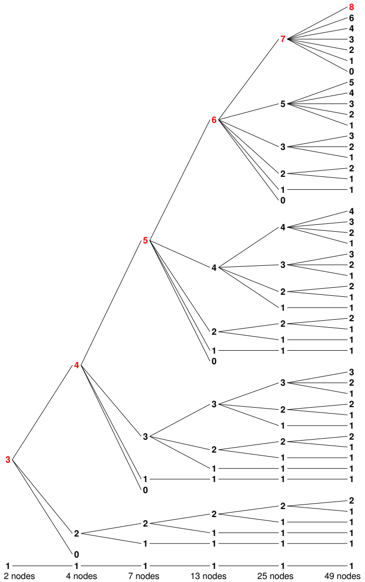
Supertree

Upper bound for the number of descendants of semigroups of genus 6



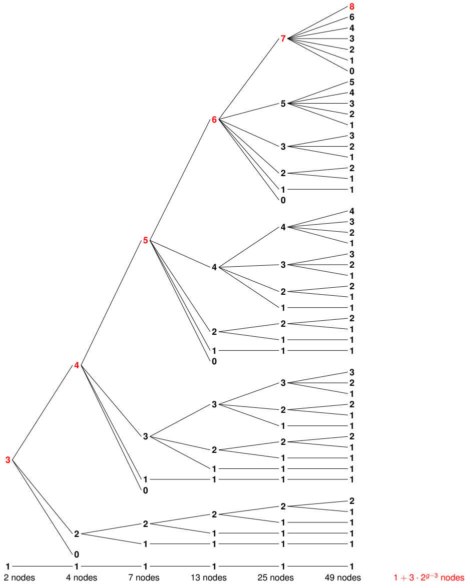
Supertree

Upper bound for the number of descendants of semigroups of genus 7



Supertree

Upper bound for the number of descendants of semigroups of genus g



Supertree

Lemma

For $g \geqslant 3$*,*

$$2F_g \leqslant n_g \leqslant 1 + 3 \cdot 2^{g-3}.$$

Bounds on n_g

g	$2F_g$	n_g	$1+3\cdot 2^{g-3}$	C_g
0		1		1
1		1		1
2	2	2		2
3	4	4	4	5
4	6	7	7	14
5	10	12	13	42
6	16	23	25	132
7	26	39	49	429
8	42	67	97	1430
9	68	118	193	4862
10	110	204	385	16796
11	178	343	769	58786
12	288	592	1537	208012
13	466	1001	3073	742900
14	754	1693	6145	2674440
15	1220	2857	12289	9694845
16	1974	4806	24577	35357670
17	3194	8045	49153	129644790
18	5168	13467	98305	477638700
19	8362	22464	196609	1767263190
20	13530	37396	393217	6564120420
21	21892	62194	786433	24466267020
22	35422	103246	1572865	91482563640
23	57314	170963	3145729	343059613650
24	92736	282828	6291457	1289904147324
25	150050	467224	12582913	4861946401452
26	242786	770832	25165825	18367353072152
27	392836	1270267	50331649	69533550916004
28	635622	2091030	100663297	263747951750360
29	1028458	3437839	201326593	1002242216651368
30	1664080	5646773	402653185	3814986502092304

Basic notions

Gaps, non-gaps, genus, Frobenius number, conductor, enumeration
Generators

Classical problems

Frobenius' coin exchange problem
Hurwitz question
Wilf's conjecture

Counting

Conjecture
Dyck paths and Catalan bounds
Semigroup tree and Fibonacci bounds
Ordinarization transform and ordinarization tree
Quasi-ordinarization transform and quasi-ordinarization forest

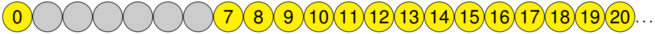
Ordinary numerical semigroups

The multiplicity of a numerical semigroup is its smallest non-zero nongap.

Ordinary numerical semigroups

The multiplicity of a numerical semigroup is its smallest non-zero nongap.

A non-trivial numerical semigroup is ordinary if m=F + 1.



Ordinarization of semigroups

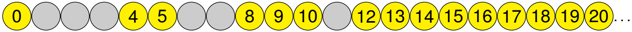
Ordinarization transform of a semigroup:

- Remove the multiplicity
- Add the Frobenius number

Ordinarization of semigroups

Ordinarization transform of a semigroup:

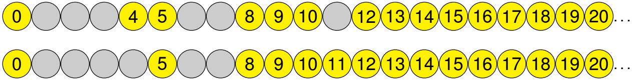
- Remove the multiplicity
- Add the Frobenius number



Ordinarization of semigroups

Ordinarization transform of a semigroup:

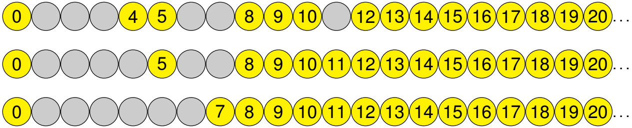
- Remove the multiplicity
- Add the Frobenius number



Ordinarization of semigroups

Ordinarization transform of a semigroup:

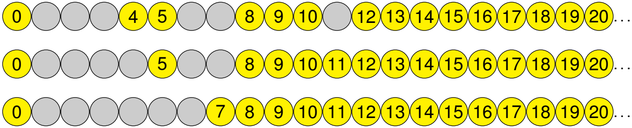
- Remove the multiplicity
- Add the Frobenius number



Ordinarization of semigroups

Ordinarization transform of a semigroup:

- Remove the multiplicity
- Add the Frobenius number

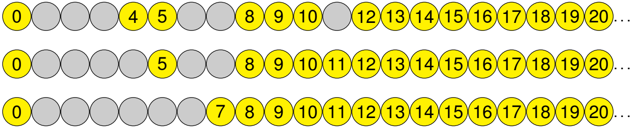


- The result is another numerical semigroup.

Ordinarization of semigroups

Ordinarization transform of a semigroup:

- Remove the multiplicity
- Add the Frobenius number

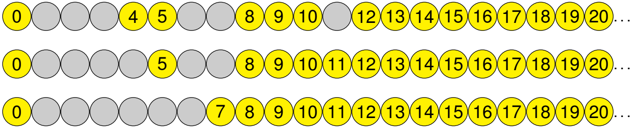


- The result is another numerical semigroup.
- The genus is kept constant in all the transforms.

Ordinarization of semigroups

Ordinarization transform of a semigroup:

- Remove the multiplicity
- Add the Frobenius number



- The result is another numerical semigroup.
- The genus is kept constant in all the transforms.
- Repeating several times (:= ordinarization number) we obtain an ordinary semigroup.

Tree $\mathcal{T}_g$ of numerical semigroups of genus g

The tree $\mathcal{T}_g$

Define a graph with

- **nodes** corresponding to semigroups of genus g
- **edges** connecting each semigroup to its ordinarization transform

$$o(\Lambda) - \Lambda$$

Tree $\mathcal{T}_g$ of numerical semigroups of genus g

The tree $\mathcal{T}_g$

Define a graph with

- **nodes** corresponding to semigroups of genus g
- **edges** connecting each semigroup to its ordinarization transform

$$o(\Lambda) - \Lambda$$

$\mathcal{T}_g$ is a tree rooted at the unique ordinary semigroup of genus g.

Tree $\mathcal{T}_g$ of numerical semigroups of genus g

The tree $\mathcal{T}_g$

Define a graph with

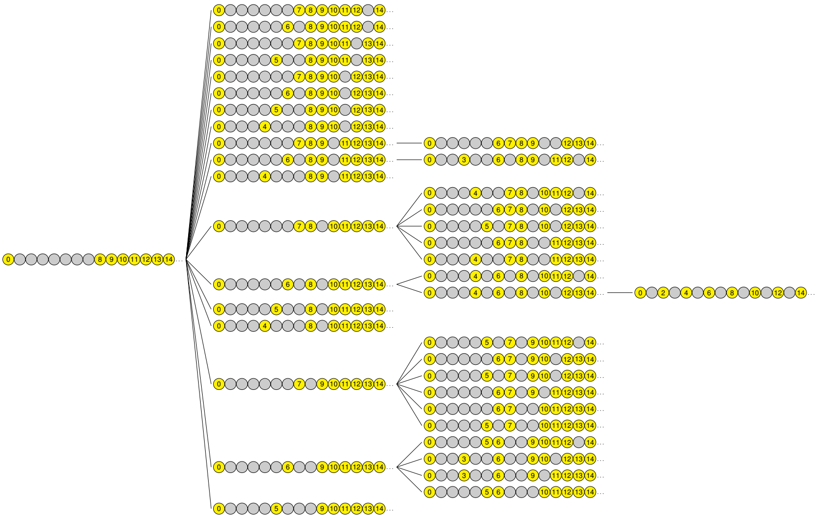
- **nodes** corresponding to semigroups of genus g
- **edges** connecting each semigroup to its ordinarization transform

$$o(\Lambda) - \Lambda$$

$\mathcal{T}_g$ is a tree rooted at the unique ordinary semigroup of genus g.

Contrary to $\mathcal{T}$, $\mathcal{T}_g$ has only a finite number of nodes (indeed, n_g).

Tree $\mathcal{T}_g$ of numerical semigroups of genus g



$\mathcal{T}_g$ and $\mathcal{T}$

Lemma

If Λ_1 is a descendant of Λ_2 in $\mathcal{T}$ then $o(\Lambda_1)$ is a descendant of $o(\Lambda_2)$ in $\mathcal{T}$.

Lemma

If Λ_1 and Λ_2 are siblings in $\mathcal{T}$ then they are siblings in $\mathcal{T}_g$.

Tree $\mathcal{T}_g$ of numerical semigroups of genus g

The depth of a semigroup of genus g in $\mathcal{T}_g$ is its ordinarization number.

Tree $\mathcal{T}_g$ of numerical semigroups of genus g

The depth of a semigroup of genus g in $\mathcal{T}_g$ is its ordinarization number.

Lemma

1. *The ordinarization number of a numerical semigroup of genus g is the number of its non-zero non-gaps which are $\leqslant g$.*
2. *The maximum ordinarization number of a semigroup of genus g is $\lfloor \frac{g}{2} \rfloor$.*
3. *The unique numerical semigroup of genus g and ordinarization number $\lfloor \frac{g}{2} \rfloor$ is $\{0, 2, 4, \dots, 2g, 2g+1, 2g+2, \dots\}$.*

Conjecture

$n_{g,r}$: number of semigroups of genus g and ordinarization number r.

Conjecture

- $n_{g,r} \leqslant n_{g+1,r}$
- Equivalently, the number of semigroups in $\mathcal{T}_g$ at a given depth is at most the number of semigroups in $\mathcal{T}_{g+1}$ at the same depth.

Conjecture

$n_{g,r}$: number of semigroups of genus g and ordinarization number r.

Conjecture

- $n_{g,r} \leqslant n_{g+1,r}$
- Equivalently, the number of semigroups in $\mathcal{T}_g$ at a given depth is at most the number of semigroups in $\mathcal{T}_{g+1}$ at the same depth.

This conjecture would prove $n_g \leqslant n_{g+1}$.

Conjecture

$n_{g,r}$: number of semigroups of genus g and ordinarization number r.

Conjecture

- $n_{g,r} \leqslant n_{g+1,r}$
- Equivalently, the number of semigroups in $\mathcal{T}_g$ at a given depth is at most the number of semigroups in $\mathcal{T}_{g+1}$ at the same depth.

This conjecture would prove $n_g \leqslant n_{g+1}$.

This result is proved for the lowest and largest depths.

Computational evidence

r\ g	g=0	g=1	g=2	g=3	g=4	g=5	g=6	g=7	g=8	g=9	g=10	g=11	g=12	g=13	g=14	g=15	g=16	g=17	g=18	g=19	g=20	g=21
r=0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
r=1			1	3	5	9	12	18	22	30	35	45	51	63	70	84	92	108	117	135	145	165
r=2					1	2	9	19	39	70	118	196	281	432	586	838	1080	1490	1835	2449	2956	3804
r=3							1	1	4	16	47	97	228	442	844	1462	2447	4017	6127	9516	13693	20152
r=4									1	1	2	3	28	60	180	442	1083	2202	4611	8579	15830	27493
r=5											1	1	2	2	9	27	93	215	721	1685	4417	9633
r=6													1	1	2	2	7	9	45	89	319	889
r=7															1	1	2	2	7	7	25	47
r=8																	1	1	2	2	7	7
r=9																			1	1	2	2
r=10																					1	1

r\ g	g=22	g=23	g=24	g=25	g=26	g=27	g=28	g=29	g=30	g=31	g=32	g=33	g=34	g=35	g=36	g=37
r=0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
r=1	176	198	210	234	247	273	287	315	330	360	376	408	425	459	477	513
r=2	4498	5690	6582	8162	9352	11370	12879	15480	17317	20569	22877	26812	29610	34454	37739	43538
r=3	27768	39726	52312	72494	93341	125600	157758	208370	255661	331626	401389	510031	608832	764927	899285	1114817
r=4	46615	76616	120795	189550	285103	429618	618555	905721	1255646	1790138	2418323	3354611	4425179	6031518	7767784	10392180
r=5	21378	41912	83951	153896	281388	487211	831654	1374366	2218771	3524257	5445975	8352388	12435320	18555615	26695019	38853706
r=6	2635	6446	17582	39214	90574	188007	394521	756910	1469758	2662254	4823002	8344482	14314198	23747986	38898550	62372773
r=7	142	340	1266	3483	10171	26489	69692	161111	382713	816457	1763299	3533977	7088495	13371197	25321828	45500820
r=8	23	24	96	157	553	1570	5281	14835	43790	113548	294908	701946	1652408	3632809	7973030	16368101
r=9	7	7	23	23	69	95	301	627	2457	7168	23475	68223	194677	512838	1323375	3178140
r=10	2	2	7	7	23	23	68	70	228	309	1142	2994	10901	33846	109619	318308
r=11	1	1	2	2	7	7	23	23	68	68	202	232	740	1249	4843	14332
r=12			1	1	2	2	7	7	23	23	68	68	200	201	649	759
r=13					1	1	2	2	7	7	23	23	68	68	200	200
r=14							1	1	2	2	7	7	23	23	68	68
r=15									1	1	2	2	7	7	23	23
r=16											1	1	2	2	7	7
r=17													1	1	2	2
r=18															1	1

r\ g	g=38	g=39	g=40	g=41	g=42	g=43	g=44	g=45	g=46	g=47	g=48	g=49
r=0	1	1	1	1	1	1	1	1	1	1	1	1
r=1	532	570	590	630	651	693	715	759	782	828	852	900
r=2	47510	54320	58986	67072	72419	81855	88142	98946	106170	118716	126844	141164
r=3	1299978	1590237	1836517	2226669	2545983	3059220	3477286	4134725	4669073	5518427	6185260	7256830
r=4	13180451	17322789	21616641	28040199	34458068	44142389	53663689	67788397	81530366	102094609	121404838	150477267
r=5	54507523	77486888	106094921	148091995	198378083	272201928	358476988	483240666	626315811	833944191	1063739070	1397557241
r=6	98298482	152816803	232801607	352797809	521753229	772496765	1114488292	1614321267	2277566111	3242295418	4478817624	6268430457
r=7	81612546	140878791	241699680	402445891	664483703	1072569052	1711738040	2688862529	4165828031	6388426599	9636305171	14462411903
r=8	33550240	65385970	126969443	235541563	436401532	777427260	1380117648	2375549463	4064063006	6774823275	11221522599	18200647631
r=9	7487630	16760501	36890000	77385799	160762381	319996692	631894288	1203245544	2273796763	4158339885	7567139870	13367227712
r=10	899807	2383461	6101724	14810797	34997273	79159902	175168573	373545010	782283651	1585487022	3171168252	6150909456
r=11	51663	164512	519339	1509557	4237829	11221868	28679326	70097864	166062233	379419480	845334246	1824208237
r=12	2527	5652	21994	71261	252707	803934	2492982	7226212	20114114	53281902	136131501	334153690
r=13	616	649	1925	2679	9947	27432	106780	361575	1245778	3945659	12053243	34718395
r=14	200	200	615	617	1800	1939	6144	11138	43824	140489	537134	1835716
r=15	68	68	200	200	615	615	1766	1804	5254	6320	22087	52194
r=16	23	23	68	68	200	200	615	615	1764	1765	5102	5278
r=17	7	7	23	23	68	68	200	200	615	615	1764	1764
r=18	2	2	7	7	23	23	68	68	200	200	615	615
r=19	1	1	2	2	7	7	23	23	68	68	200	200
r=20			1	1	2	2	7	7	23	23	68	68
r=21					1	1	2	2	7	7	23	23
r=22							1	1	2	2	7	7
r=23									1	1	2	2
r=24											1	1

Lemma (Bernardini and Torres (2017))

The sequence f_γ given by

$$
\begin{aligned}
f_0 &= 1,\\
f_1 &= 2,\\
f_2 &= 7,\\
f_3 &= 23,\\
f_4 &= 68,\\
f_5 &= 200,\\
f_6 &= 615,\\
f_7 &= 1764,\\
f_8 &= 5060,\\
f_9 &= 14626,\\
&\ldots
\end{aligned}
$$

also counts the number of semigroups of genus 3γ and γ even gaps.

Conjecture (Bernardini,Torres)

$$f_\gamma \sim \varphi^{2\gamma}$$

Basic notions

- Gaps, non-gaps, genus, Frobenius number, conductor, enumeration
- Generators

Classical problems

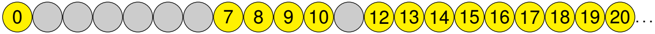
- Frobenius' coin exchange problem
- Hurwitz question
- Wilf's conjecture

Counting

- Conjecture
- Dyck paths and Catalan bounds
- Semigroup tree and Fibonacci bounds
- Ordinarization transform and ordinarization tree
- Quasi-ordinarization transform and quasi-ordinarization forest

Quasi-ordinary numerical semigroups

A non-ordinary semigroup Λ is a quasi-ordinary semigroup if $\Lambda \cup F$ is ordinary.



Quasi-ordinarization of semigroups

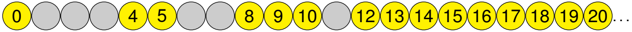
Quasi-ordinarization transform of a non-ordinary semigroup:

- Remove the multiplicity
- Add the second largest gap

Quasi-ordinarization of semigroups

Quasi-ordinarization transform of a non-ordinary semigroup:

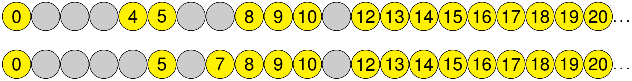
- Remove the multiplicity
- Add the second largest gap



Quasi-ordinarization of semigroups

Quasi-ordinarization transform of a non-ordinary semigroup:

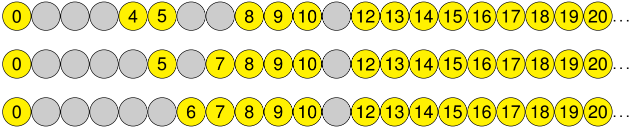
- Remove the multiplicity
- Add the second largest gap



Quasi-ordinarization of semigroups

Quasi-ordinarization transform of a non-ordinary semigroup:

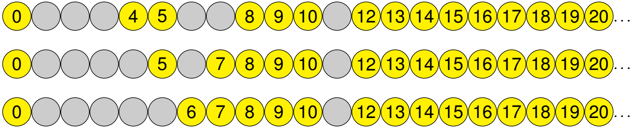
- Remove the multiplicity
- Add the second largest gap



Quasi-ordinarization of semigroups

Quasi-ordinarization transform of a non-ordinary semigroup:

- Remove the multiplicity
- Add the second largest gap

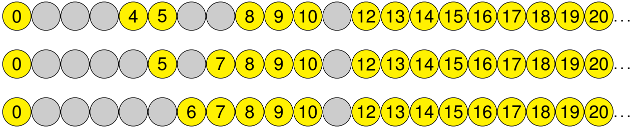


- The result is another numerical semigroup.

Quasi-ordinarization of semigroups

Quasi-ordinarization transform of a non-ordinary semigroup:

- Remove the multiplicity
- Add the second largest gap

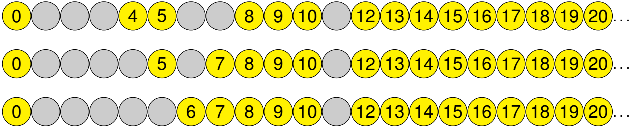


- The result is another numerical semigroup.
- The genus is kept constant in all the transforms.

Quasi-ordinarization of semigroups

Quasi-ordinarization transform of a non-ordinary semigroup:

- Remove the multiplicity
- Add the second largest gap

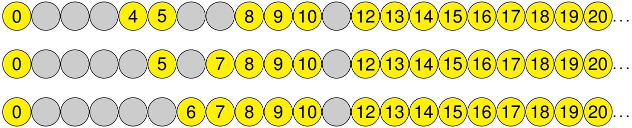


- The result is another numerical semigroup.
- The genus is kept constant in all the transforms.
- Repeating several times (:= quasi-ordinarization number) we obtain a quasi-ordinary semigroup.

Quasi-ordinarization of semigroups

Quasi-ordinarization transform of a non-ordinary semigroup:

- Remove the multiplicity
- Add the second largest gap



- The result is another numerical semigroup.
- The genus is kept constant in all the transforms.
- Repeating several times (:= quasi-ordinarization number) we obtain a quasi-ordinary semigroup.

Quasi-ordinarization transform of an ordinary semigroup is defined to be itself.

Forest $\mathcal{F}_g$ of numerical semigroups of genus g

The forest $\mathcal{F}_g$

Define a graph with

- **nodes** corresponding to semigroups of genus g
- **edges** connecting each semigroup to its quasi-ordinarization transform

$$q(\Lambda) - \Lambda$$

Forest $\mathcal{F}_g$ of numerical semigroups of genus g

The forest $\mathcal{F}_g$

Define a graph with

- **nodes** corresponding to semigroups of genus g
- **edges** connecting each semigroup to its quasi-ordinarization transform

$$q(\Lambda) - \Lambda$$

$\mathcal{F}_g$ is a forest with roots at the quasi-ordinary semigroups of genus g, and the unique ordinary semigroup of genus g.

Forest $\mathcal{F}_g$ of numerical semigroups of genus g

The forest $\mathcal{F}_g$

Define a graph with

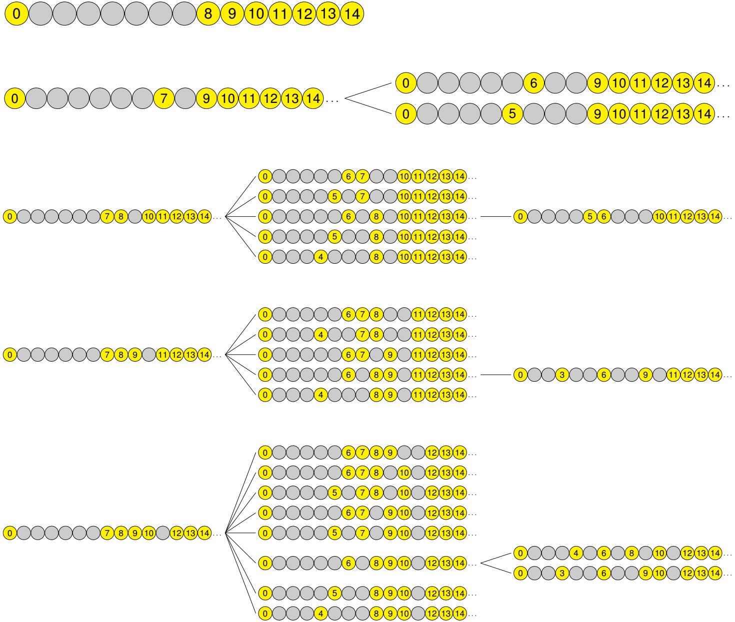
- **nodes** corresponding to semigroups of genus g
- **edges** connecting each semigroup to its quasi-ordinarization transform

$$q(\Lambda) - \Lambda$$

$\mathcal{F}_g$ is a forest with roots at the quasi-ordinary semigroups of genus g, and the unique ordinary semigroup of genus g.

Contrary to $\mathcal{T}_g$, $\mathcal{F}_g$ is a forest.

Forest $\mathcal{F}_g$ of numerical semigroups of genus g



$\mathcal{F}_g$ and $\mathcal{T}$

Lemma

If Λ_1 is descendant of Λ_2 in $\mathcal{T}$ then $q(\Lambda_1)$ is a niece/nephew of $q(\Lambda_2)$ in $\mathcal{T}$.

Lemma

If Λ_1 and Λ_2 are siblings in $\mathcal{T}$ then they are siblings in $\mathcal{T}_g$ but not in $\mathcal{F}_g$.

Lemma

If Λ_1 and Λ_2 are siblings in $\mathcal{T}_g$ then $q(\Lambda_1)$ and $q(\Lambda_2)$ are siblings in $\mathcal{T}$.

Lemma

If Λ_1 and Λ_2 are siblings in $\mathcal{T}$ then $q(\Lambda_1)$ and $q(\Lambda_2)$ are siblings in $\mathcal{T}$.

Further contributions on counting

Maria Bras-Amoróos and Anna de Mier. Representation of numerical semigroups by Dyck paths. Semigroup Forum, 75(3):677-682, 2007.

Maria Bras-Amorós. Fibonacci-like behavior of the number of numerical semigroups of a given genus. Semigroup Forum, 76(2):379-384, 2008.

Maria Bras-Amorós. Bounds on the number of numerical semigroups of a given genus. J. Pure Appl. Algebra, 213(6):997-1001, 2009.

Maria Bras-Amorós and Stanislav Bulygin. Towards a better understanding of the semigroup tree. Semigroup Forum, 79(3):561-574, 2009.

Sergi Elizalde. Improved bounds on the number of numerical semigroups of a given genus. Journal of Pure and Applied Algebra, 214:1404-1409, 2010.

Yufei Zhao. Constructing numerical semigroups of a given genus. Semigroup Forum, 80(2):242-254, 2010.

Víctor Blanco, Pedro A. García-Sánchez, and Justo Puerto. Counting numerical semigroups with short generating functions, Internat. J. Algebra Comput. 21:1217–1235, 2011.

Nathan Kaplan. Counting numerical semigroups by genus and some cases of a question of Wilf, J. Pure Appl. Algebra 216: 1016–1032, 2012.

Maria Bras-Amorós. The ordinarization transform of a numerical semigroup and semigroups with a large number of intervals, J. Pure Appl. Algebra 216:2507–2518, 2012.

Evan O'Dorney. Degree asymptotics of the numerical semigroup tree, Semigroup Forum 87:601–616, 2013.

Matheus Bernardini and Fernando Torres. Counting numerical semigroups by genus and even gaps, Discrete Mathematics 340 (12):2853-2863, 2017.

Nathan Kaplan. Counting numerical semigroups, Amer. Math. Monthly 124: 862-875, 2017.